EXERCISE 4.1

1. Evaluate the following determinants :

(i) $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$ (ii) $\begin{vmatrix} \cos 90° & -\cos 45° \\ \sin 90° & \sin 45° \end{vmatrix}$.

2. Evaluate the following determinants :

(i) $\begin{vmatrix} -1 & 31 & 40 \\ 0 & 5 & -432 \\ 0 & 0 & 20 \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 0 & 0 \\ -25 & 3 & 0 \\ 71 & 64 & 7 \end{vmatrix}$.

3. (i) If $A = \begin{vmatrix} 5 & 6 & -3 \\ -4 & 3 & 2 \\ -4 & -7 & 3 \end{vmatrix}$, then write the cofactor of the element a_{21} of its 2nd row.

(ii) If A_{ij} is the cofactor of the element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then write the value of $a_{32} A_{32}$.

4. Find the value of x if

(i) $\begin{vmatrix} 2x+5 & 3 \\ 5x+2 & 9 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} x+2 & 3 \\ x+5 & 4 \end{vmatrix} = 3$.

5. (i) If $x \in$ N and $\begin{vmatrix} x & 3 \\ 4 & x \end{vmatrix} = \begin{vmatrix} 4 & -3 \\ 0 & 1 \end{vmatrix}$, find the value(s) of x.

(ii) If $x \in$ I and $\begin{vmatrix} 2x & 3 \\ -1 & x \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ x & 3 \end{vmatrix}$, find the value(s) of x.

6. There are two real value(s) of x, for which the value of the determinant $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & x & -1 \\ 0 & 4 & 2x \end{vmatrix}$ is 86.

Find the values of x.

7. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$.

8. Evaluate the following determinants:

(i) $\begin{vmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$.

9. Prove that $\begin{vmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$.

Answers

1. (i) $x^3 - x^2 + 2$ (ii) $\frac{1}{\sqrt{2}}$
2. (i) -100 (ii) 42
3. (i) 3 (ii) 110
4. (i) -13 (ii) 10
5. (i) 4 (ii) -2
6. 3, -7
8. (i) 0 (ii) 46

EXERCISE 4.2

1. Without expanding, find the values of :

(i) $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$ (iii) $\begin{vmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{vmatrix}$.

Hint. (ii) Operate $C_3 \to C_3 - 9C_2$.
(iii) Operate $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_2$.

2. Without expanding, find the values of :

(i) $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$ (ii) $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$.

3. Without expanding, find the values of :

(i) $\begin{vmatrix} a-b & b-c & c-a \\ x-y & y-z & z-x \\ p-q & q-r & r-p \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$.

4. Without expanding, find the values of:

(i) $\begin{vmatrix} \frac{1}{a} & 1 & bc \\ \frac{1}{b} & 1 & ca \\ \frac{1}{c} & 1 & ab \end{vmatrix}$ (ii) $\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix}$.

Hint. (i) Operate $C_1 \to abc\, C_1$
(ii) Operate $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_2$

5. If A is a square matrix of order 2 and $|A| = -5$, find the value of $|3A|$.
6. If A is a square matrix of order 3 and $|A| = 4$, then write the value of $|2A|$.
7. If A is a square matrix of order 3 and $|A| = -2$, find the value of $|-5A|$.
8. If A is a square matrix of order 3 and $|3A| = k|A|$, then find the value of k.
9. If A is a square matrix such that $|A| = 5$, write the value of $|AA^T|$.

10. If A is a square matrix satisfying $AA' = I$, write the value of $|A|$.

11. If $A = \begin{bmatrix} x & 2 \\ 2 & x \end{bmatrix}$ and $|A^3| = 125$, find the value(s) of x.

12. Show that one root of the equation $\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$ is $-(a+b+c)$.

13. Without expanding, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

14. Use properties of determinants to solve for x :

(i) $\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0$ (ii) $\begin{vmatrix} 3-x & -1 & 1 \\ -1 & 5-x & -1 \\ 1 & -1 & 3-x \end{vmatrix} = 0$.

15. Using the properties of determinants, prove that

$\begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix} = 0$, where α, β and γ are in A.P.

16. Using properties of determinants, prove that :

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

Hint. Operate $R_1 \to R_1 - R_2 - R_3$ and take (-2) out from new R_1.

Using properties of determinants, prove the following (17 to 24) identities:

17. (i) $\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = 1+a+b+c$

(ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$.

18. (i) $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$.

(ii) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$.

19. (i) $\begin{vmatrix} x & y & 1 \\ \alpha & x & 1 \\ \alpha & \beta & 1 \end{vmatrix} = (x-\alpha)(x-\beta)$ (ii) $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix} = xy$.

20. $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ca & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2.$

21. $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$

22. (*i*) $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a).$

(*ii*) $\begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix} = (x-p)(x^2+px-2q^2).$

Hint. (*i*) Operate $C_1 \to C_1 + C_3, C_2 \to C_2 + C_3$.

23. $\begin{vmatrix} y+z & x+y & x \\ z+x & y+z & y \\ x+y & z+x & z \end{vmatrix} = x^3 + y^3 + z^3 - 3xyz.$

Hint. Operate $C_1 \to C_1 + C_3, C_2 \to C_2 - C_3$.

24. $\begin{vmatrix} a & b-c & c-b \\ a-c & b & c-a \\ a-b & b-a & c \end{vmatrix} = (a+b-c)(b+c-a)(c+a-b).$

Hint. Operate $C_1 \to C_1 + C_2, C_2 \to C_2 + C_3$.

25. Using properties of determinants, solve the following equations for x :

(*i*) $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$ (*ii*) $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0.$

Answers

1. (*i*) 0	(*ii*) 0	(*iii*) 0	2. (*i*) 0	(*ii*) 0	
3. (*i*) 0	(*ii*) 0	4. (*i*) 0	(*ii*) 0	5. −45	6. 32
7. 250	8. 27	9. 25	10. 1, −1	11. 3, −3	
14. (*i*) 0, 0, − (a + b + c)		(*ii*) 2, 3, 6	25. (*i*) $\frac{2}{3}, \frac{11}{3}, \frac{11}{3}$	(*ii*) 1, 1, −9	

AREA OF A TRIANGLE

We know that the area of a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$

$= \text{the absolute value of } \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3).$

Also $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1(y_2 - y_3) - y_1(x_2 - x_3) + 1(x_2y_3 - x_3y_2)$ (Expansion by R_1)

$= x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3.$

EXERCISE 4.4

1. For what value of x, is the following matrix singular?

$$\begin{bmatrix} 3-2x & x+1 \\ 2 & 4 \end{bmatrix}$$

2. Which of the following matrices are non-singular?

(i) $\begin{bmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix}$.

3. Write the adjoint of the matrices :

(i) $\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 0 \\ -2 & -5 \end{bmatrix}$.

4. (i) For what value of k, the matrix $\begin{bmatrix} 2 & k \\ 3 & 5 \end{bmatrix}$ has no inverse?

(ii) If $A = \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$, then write A^{-1}.

(*iii*) Write the inverse of the matrix $\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$.

5. (*i*) If $A = \begin{bmatrix} 8 & 2 \\ 3 & 2 \end{bmatrix}$, then find $|adj\ A|$.

(*ii*) If A is a square matrix of order 3 and $|A| = 5$, then find $|adj\ A|$.

(*iii*) If A is a non-singular square matrix of order 3 such that $|adj\ A| = 64$, find $|A|$.

(*iv*) If A is a square matrix of order 3 such that $|A| = -2$, then find $|3\ adj\ A|$.

(*v*) If $|A^{-1}| = 5$, then find the value of $|A|$.

(*vi*) If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that $A^{-1} = kA$, then find the value of k.

(*vii*) If $A\,(adj\ A) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then find the value of $|adj\ A|$.

6. Without computing *adj* A, find $|adj\ A|$ if

(*i*) $A = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} -2 & 0 & 0 \\ 3 & 4 & 0 \\ 10 & -7 & 3 \end{bmatrix}$.

7. (*i*) If $A = \begin{bmatrix} 2 & 1 \\ 7 & 5 \end{bmatrix}$, find $|A\ adj\ A|$.

(*ii*) If $A = \begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}$, find *adj* (*adj* A).

8. (*i*) If A is a non-singular matrix, then show that $|A^{-1}| = \frac{1}{|A|}$.

(*ii*) If A is a matrix such that $A^3 = I$, then show that A is invertible.

(*iii*) If A is a matrix such that $A^4 = I$, then show that $A^{-1} = A^3$.

9. If $A = \begin{bmatrix} 3 & -5 \\ 4 & 2 \end{bmatrix}$, find A(*adj* A).

10. If the matrix $\begin{bmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{bmatrix}$ is singular, find x.

11. Find the adjoint of the following matrices:

(*i*) $A = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$.

Also verify that $A\,(adj\ A) = |A|\,I_2 = (adj\ A)\,A$.

12. If $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, find A (*adj* A) without calculating *adj* A.

13. Find the adjoint of the following matrices:

(*i*) $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$.

Also verify that $A\,(adj\ A) = |A|\,I_3 = (adj\ A)\,A$.

14. (*i*) Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$.

(*ii*) If $A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$, show that $A - 3I = 2(I + 3A^{-1})$.

15. If $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & x \\ 2 & 3 & 1 \end{bmatrix}$ and $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -8 & 5 \\ -1 & 6 & -3 \\ 1 & 2y & 1 \end{bmatrix}$, find the value(s) of x and y.

Hint. We know that $A^{-1} A = I$.

16. If $A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 4 & 0 \end{bmatrix}$, verify that $adj\,(AB) = (adj\,B)\,(adj\,A)$.

17. Find the inverse of each of the following matrices:

(*i*) $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$.

Also verify that $A^{-1} A = I_3$.

18. (*i*) Show that $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ satisfies the equation $A^2 - 3A - 7I = O$ and hence, find A^{-1}.

(*ii*) If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, then find the value of λ so that $A^2 = \lambda A - 2I$. Hence, find A^{-1}.

19. (*i*) Find a 2×2 matrix B such that $B\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$.

(*ii*) Solve the matrix equation $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$.

Answers

1. 1 2. Matrix in (*ii*) 3. (*i*) $\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -5 & 0 \\ 2 & 3 \end{bmatrix}$

4. (*i*) $\frac{10}{3}$ (*ii*) $\begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$ (*iii*) $\frac{1}{7}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

5. (*i*) 10 (*ii*) 25 (*iii*) 8, –8 (*iv*) 108 (*v*) $\frac{1}{5}$ (*vi*) $\frac{1}{19}$ (*vii*) 9

6. (*i*) 7 (*ii*) 576 7. (*i*) 9 (*ii*) $\begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 26 & 0 \\ 0 & 26 \end{bmatrix}$ 10. $-\frac{4}{3}$ 11. (*i*) $\begin{bmatrix} 4 & -1 \\ -3 & -2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ 13. (*i*) $\begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$

14. (*i*) $\frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ 15. $x = 1, y = -1$