11

ISAAC NEWTON

Limit, Continuity and Differentiability

MEANING OF $x \to a$

$x \to a$ is read as 'x tends to a' or 'x approaches a', where x is a variable. It can be changed so that its value comes nearer and nearer to a, $0 < |x - a|$, where (i) $x \neq a$ and (ii) $|x - a|$ becomes smaller and smaller as we please.

LIMIT

We say that $\lim_{x \to a} f(x) = l$, if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - l| < \varepsilon$, whenever $|x - a| < \delta$. This means, smaller is the difference between x and a, smaller will be the difference between $f(x)$ and l.

INDETERMINATE FORMS

If a function $f(x)$ takes any of the following forms when $x = a$, then we say that $f(x)$ is indeterminate at $x = a$.

1. $0/0$
2. ∞/∞
3. $\infty - \infty$
4. $0 \times \infty$
5. 1^{∞}
6. 0^0
7. ∞^0

ALGEBRA OF LIMITS

1. $\lim\limits_{x\to a}[f(x)+g(x)]=\lim\limits_{x\to a}f(x)+\lim\limits_{x\to a}g(x)$
2. $\lim\limits_{x\to a}[f(x)-g(x)]=\lim\limits_{x\to a}f(x)-\lim\limits_{x\to a}g(x)$
3. $\lim\limits_{x\to a}[kf(x)]=k\lim\limits_{x\to a}f(x)$
4. $\lim\limits_{x\to a}[f(x)\cdot g(x)]=k\left[\lim\limits_{x\to a}f(x)\right].\left[\lim\limits_{x\to a}g(x)\right]$
5. $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}$, when $\lim\limits_{x\to a}g(x)\neq 0$
6. $\lim\limits_{x\to a}[f(x)]^n=\left[\lim\limits_{x\to a}f(x)\right]^n$
7. $\lim\limits_{x\to a}e^{f(x)}=e^{\lim\limits_{x\to a}f(x)}$
8. $\lim\limits_{x\to a}[f(x)]^{g(x)}=e^{\left[\lim\limits_{x\to a}g(x)\log_e f(x)\right]}$
9. $\log\left[\lim\limits_{x\to a}f(x)\right]=\lim\limits_{x\to a}[\log f(x)]$, when $\lim\limits_{x\to a}f(x)>0$

ONE-SIDED LIMITS

This method is applied to find the limit at $x = a$ when the function is defined differently for $x > a$, $x = a$ and $x < a$.

RIGHT-HAND LIMIT

We say that the right-hand limit of $f(x)$ at $x = a$ is A if $f(x)\to A$ when $x\to a$ through values greater than a, and we write

$$\lim_{x\to a^+}f(x)=A \quad\text{or}\quad \lim_{x\to a+0}f(x)=A \quad\text{or}\quad f(a+0)=A$$

WORKING RULE FOR FINDING $\lim\limits_{x\to a^+}f(x)$

Replace x by $(a + h)$ and take the limit as $h\to 0$, i.e. $\lim\limits_{x\to a^+}f(x)=\lim\limits_{h\to 0}f(a+h)$.

LEFT-HAND LIMIT

We say that the left-hand limit of $f(x)$ at $x = a$ is B if $f(x)\to B$ when $x\to a$ through values less than a, and we write

$$\lim_{x\to a^-}f(x)=B \text{ or } \lim_{x\to a-0}f(x)=B \text{ or } f(a-0)=B$$

WORKING RULE FOR FINDING $\lim\limits_{x\to a^-}f(x)$

Replace x by $(a - h)$ and take the limit as $h\to 0$, i.e. $\lim\limits_{x\to a^-}f(x)=\lim\limits_{h\to 0}f(a-h)$.

LIMIT OF A FUNCTION DERIVED FROM ONE-SIDED LIMITS

We say that $\lim\limits_{x\to a}f(x)=l$ if $\lim\limits_{x\to a^+}f(x)=\lim\limits_{x\to a^-}f(x)=l$. However, if $\lim\limits_{x\to a^+}f(x)\neq\lim\limits_{x\to a^-}f(x)$ or if any of the limits $\lim\limits_{x\to a^+}f(x)$ or $\lim\limits_{x\to a^-}f(x)$ does not exist, then we say that $\lim\limits_{x\to a}f(x)$ does not exist.

Illustration 1. Evaluate $\lim\limits_{x\to 1}\dfrac{x^3-x^2\log x+\log x-1}{x^2-1}$.

Solution: The given limit

$$=\lim_{x\to 1}\frac{(x^3-1)-(x^2-1)\log x}{x^2-1}$$

$$=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)-(x-1)(x+1)\log x}{(x-1)(x+1)}$$

$$=\lim_{x\to 1}\frac{(x-1)[x^2+x+1-(x+1)\log x]}{(x-1)(x+1)}$$

$$=\frac{1^2+1+1-(1+1)\log 1}{(1+1)}=\frac{3}{2}.$$

SOME IMPORTANT LIMITS

1. If $f(x)$ is a polynomial, then $\lim\limits_{x\to a}f(x)=f(a)$
2. If $a\neq 0$ and $n\in Q$, then $\lim\limits_{x\to a}\left(\dfrac{x^n-a^n}{x-a}\right)=na^{n-1}$
3. $\lim\limits_{x\to a}\left(\dfrac{x^m-a^m}{x^n-a^n}\right)=\dfrac{m}{n}a^{m-n}$
4. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1=\lim\limits_{x\to 0}\dfrac{x}{\sin x}$
5. $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=1=\lim\limits_{x\to 0}\dfrac{x}{\tan x}$
6. $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x}{x}=1=\lim\limits_{x\to 0}\dfrac{x}{\sin^{-1}x}$
7. $\lim\limits_{x\to 0}\dfrac{\tan^{-1}x}{x}=1=\lim\limits_{x\to 0}\dfrac{x}{\tan^{-1}x}$
8. $\lim\limits_{x\to 0}\dfrac{\sin x^0}{x}=\dfrac{\pi}{180}$
9. $\lim\limits_{x\to 0}\cos x=1$
10. $\lim\limits_{x\to a}\dfrac{\sin(x-a)}{x-a}=1$
11. $\lim\limits_{x\to a}\dfrac{\tan(x-a)}{x-a}=1$
12. $\lim\limits_{x\to a}\sin^{-1}x=\sin^{-1}a, |a|\leq 1$

13. $\lim_{x\to a} \cos^{-1} x = \cos^{-1} a, |a| \le 1$

14. $\lim_{x\to a} \tan^{-1} x = \tan^{-1} a, -\infty < a < \infty$

L'HÔPITAL'S RULE

If $\frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when $x = a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

This is known as **L'Hôpital's rule.**

SOME IMPORTANT EXPANSIONS THAT ARE USEFUL FOR FINDING LIMITS

1. For $|y| < 1$, $(1+x)^n$

$$= \left[1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 \ldots\right]$$

2. $\left(\frac{x^n - a^n}{x-a}\right) = (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-1})$

3. $e^x = \left[1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots + \frac{x^n}{\underline{|n}} + \cdots\right]$

4. $a^x = \left[1 + x(\log x) + \frac{x^2}{\underline{|2}}(\log a)^2 + \cdots\right]$

5. $\log(1+x) = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right]$

6. $\sin x = \left[x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots\right]$

7. $\cos x = \left[1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \cdots\right]$

8. $\tan x = \left[x - \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\right]$

9. $\tan^{-1} x = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right]$

10. $e^{-x} = \left[1 - x + \frac{x^2}{\underline{|2}} - \frac{x^3}{\underline{|3}} + \cdots\right]$

11. $\log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right]$

12. $\sec x = \left[1 + \frac{x^2}{\underline{|2}} + \frac{5x^4}{\underline{|4}} + \cdots\right]$

13. $\sin^{-1} x = \left[x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \cdots\right]$

LOGARITHMIC LIMITS

We use the series $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \infty$, where $-1 < x \le 1$ and expansion is true only if base is e.

1. $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

2. $\lim_{x\to e} \log_e x = 1$

3. $\lim_{x\to 0} \frac{\log(1-x)}{x} = -1$

4. $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e; a > 0, \ne 1$

EXPONENTIAL LIMITS

Based on series expansion, we use the series

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots \infty$$

1. $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

2. $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$

3. $\lim_{x\to 0} \frac{e^{\lambda x} - 1}{x} = \lambda \ (\lambda \ne 0)$

BASED ON THE FORM 1^∞

To evaluate the exponential form 1^∞, we use the following results:

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} [1 + f(x)]^{1/g(x)} = e^{\lim_{x\to a} \frac{f(x)}{g(x)}} \text{ or}$$

when $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} [f(x)]^{g(x)} = \left[\lim_{x\to a} f(x)\right]^{\lim_{x\to a} g(x)}$$

1. $\lim_{x\to 0} (1+x)^{1/x} = e$

2. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$

3. $\lim_{x\to 0}(1+\lambda x)^{1/x} = e^{\lambda}$

4. $\lim_{x\to\infty}\left(1+\frac{\lambda}{x}\right)^x = e^{\lambda}$

5. $\lim_{x\to\infty} a^x = \begin{cases}\infty, & \text{if } a>1\\ 0, & \text{if } a<1\end{cases}$

Illustration 2. Evaluate $\lim_{x\to\pm\infty} x(\sqrt{x^2+k}-x),\ k>0$.

Solution: $\lim_{x\to\pm\infty} x(\sqrt{x^2+k}-x)\frac{(\sqrt{x^2+k}+x)}{(\sqrt{x^2+k}+x)}$

$$= \lim_{x\to\pm\infty}\frac{x(x^2+k-x^2)}{(\sqrt{x^2+k}+x)}$$

$$= \lim_{x\to\pm\infty}\frac{xk}{\left[|x|\sqrt{\left(1+\frac{k}{x^2}\right)}\right]+x}$$

Here we have to consider two cases:

(i) When $x\to\infty; |x| = -x$, then we have

$$\lim_{x\to\infty}\frac{xk}{x\sqrt{\left(1+\frac{k}{x^2}\right)}+x} = \lim_{x\to\infty}\frac{xk}{x\left[\sqrt{\left(1+\frac{k}{x^2}\right)}+1\right]} = \frac{k}{2}$$

(ii) When $x\to-\infty; |x| = -x$, then we have

$$\lim_{x\to-\infty}\frac{xk}{-x\sqrt{\left(1+\frac{k}{x^2}\right)}+x} = \lim_{x\to-\infty}\frac{xk}{x\left[-\sqrt{\left(1+\frac{k}{x^2}\right)}+1\right]}$$

$$= \frac{k}{-1^-+1} = \frac{k}{0^-} \to -\infty.$$

TO FIND $\lim_{x\to\infty} f(x)$

Replace x by $1/y$ and take the limit as $y\to 0$.

Some Limits

1. If $|x|<1$, then $\lim_{n\to\infty} x^n = \infty$

2. If $x>1$, then $\lim_{n\to\infty} x^n = \infty$

3. $\lim_{x\to\infty} e^x = \infty$

4. $\lim_{x\to\infty} e^{-x} = 0$

5. $\lim_{x\to\infty} \log x = \infty$

6. $\lim_{n\to\infty}\frac{1}{n} = 0$ and $\lim_{n\to\infty}\frac{1}{n^2} = 0$

7. $\lim_{n\to\infty} x^{1/n} = 1$

8. $\lim_{x\to\infty}\frac{\sin x}{x} = \lim_{x\to\infty}\frac{\cos x}{x}$

9. $\lim_{x\to\infty}\frac{\sin 1/x}{1/x} = 1$

USEFUL TRIGONOMETRIC RESULTS

1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
2. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
3. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
4. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
5. $\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$
6. $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$
7. $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$
8. $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$
9. $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$
10. $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

CONTINUITY AT A POINT

A function $f(x)$ is said to be continuous at $x = a$ if $\lim_{x\to a^-} f(a) = \lim_{x\to a^+} f(x) = f(a)$, i.e. LHL = RHL = value of the function at a, i.e. $\lim_{x\to a} f(x) = f(a)$. If $f(x)$ is not continuous at $x = a$, we say that $f(x)$ is discontinuous at $x = a$.

Illustration 3. Discuss the continuity of the function [cos x] at $x = (\pi/2)$, where [ ] denotes the greatest integer function.

Solution: LHL = $\lim_{x\to(\pi/2)^-}[\cos x] = 0$

RHL = $\lim_{x\to(\pi/2)^+}[\cos x] = -1$

$$f\left(\frac{\pi}{2}\right) = \left[\cos\frac{\pi}{2}\right] = 0$$

Clearly, LHL ≠ RHL.
So, the function is discontinuous at $x = (\pi/2)$.

CONTINUITY IN AN OPEN INTERVAL

A function $f(x)$ is said to be continuous in an open interval (a, b) if it is continuous at each and every point of (a, b), i.e. $y = [x]$ is continuous in (1, 2).

CONTINUITY IN A CLOSED INTERVAL

A function $f(x)$ is said to be continuous in a closed interval $[a, b]$ if

(i) it is continuous in (a, b).

(ii) the value of the function at b is equal to the left-hand limit at b, i.e. $f(b) = \lim_{x \to b^+} f(x)$.

(iii) the value of the function at a is equal to the right-hand limit at a, i.e. $f(a) = \lim_{x \to a^+} f(x)$.

Illustration 4. Check the continuity of the function $f(x) = [x^2] - [x]^2$ for all $x \in \mathbb{R}$ at the end points of the interval $[-1, 0]$, where [ ] denotes the greatest integer function.

Solution: Continuity at $x = -1$

$$f(-1) = [(-1)^2] - [-1]^2 = [1] - (-1)^2 = 1 - 1 = 0$$

$$\text{RHL} = \lim_{x \to -1^+} \{[x^2] - [x]^2\} = 0 - 1 = -1$$

So, $f(-1) \neq \text{RHL}$

Continuity at $x = 0$

$$f(0) = [(0)^2] - [0]^2 = 0 - 0 = 0$$

$$\text{LHL} = \lim_{x \to 0^+} \{[x^2] - [x]^2\} = 0 - 1 = -1$$

So, $f(0) \neq$ LHL

Hence, the function is not continuous at the end points of the interval $[-1, 0]$.

PROPERTIES OF CONTINUOUS FUNCTIONS

Let $f(x)$ and $g(x)$ be continuous functions at $x = a$. Then,

(i) $cf(x)$ is continuous at $x = a$, where c is any constant.

(ii) $f(x) \pm g(x)$ is continuous at $x = a$.

(iii) $f(x) \cdot g(x)$ is continuous at $x = a$.

(iv) $f(x)/g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$.

(v) If $f(x)$ is continuous on $[a, b]$, such that $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one solution of the equation $f(x) = 0$ in the open interval (a, b).

DISCONTINUOUS FUNCTION

If $f(x)$ is not continuous at $x = a$, then $f(x)$ is said to be discontinuous at $x = a$ and this point is called a point of discontinuity.

TYPES OF DISCONTINUITY

Removable Discontinuity

A function f is said to possess removable discontinuity if at $x = a$,

$$L = R \neq V$$

$$\Rightarrow \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$$

i.e. the left-hand limit and the right-hand limit at $x = a$ exist and are equal, but they are not equal to the value of the function at $x = a$. This is shown in Fig. 11.1.

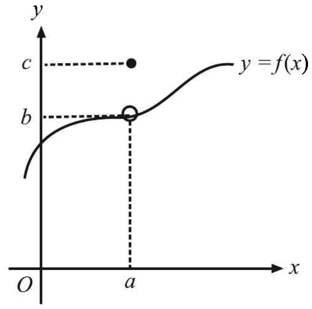

Fig. 11.1

Irremovable Discontinuity

A function f is said to possess irremovable discontinuity if at $x = a$, the left-hand limit is not equal to the right-hand limit, i.e. $L \neq R$.

$$\Rightarrow \quad \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

Discontinuity of First Kind

A function f is said to possess discontinuity of first kind at $x = a$ if at $x = a$, both the left-hand limit and the right-hand limit exist finitely but are unequal.

This can be illustrated with the help of Fig. 11.2.

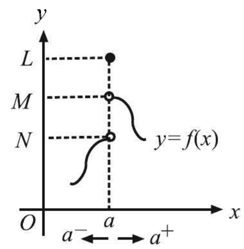

Fig. 11.2

Here, $\lim_{x \to a^-} f(x) = N$ and $\lim_{x \to a^+} f(x) = M$. Also, $f(a) = L$.

Clearly, from the figure, $N \neq M$.

Discontinuity of Second Kind

A function f is said to possess discontinuity of second kind at $x = a$ if at $x = a$, both the left-hand limit and the right-hand limit do not exist and are infinite. This can be illustrated with the help of Fig. 11.3.

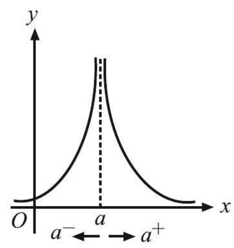

Fig. 11.3

DIFFERENTIABILITY

Let $f(x)$ be a real-valued function defined on an interval (a, b) and $x_1 \in (a, b)$. Then the function $f(x)$ is said to be *differentiable* (or *derivable*) at x_1 if

$$\lim_{h\to 0}\frac{f(x_1+h)-f(x_1)}{(x_1+h)-(x_1)} = \frac{\text{small change in } y}{\text{small change in } x}$$

or equivalently, $\lim_{x\to x_1}\frac{f(x)-f(x_1)}{x-x_1}$.

The value of the limit is denoted by $f'(x_1)$ or by $Df(x_1)$ and is usually called *derivative* of $f(x)$.

TYPES OF DERIVATIVES

Left-hand Derivatives

Regressive derivative or left-hand derivative of $f(x)$ at $x = x_1$ is given by

$$\text{LHD} = Lf'(x_1) = \lim_{h\to 0}\frac{f(x_1-h)-f(x_1)}{(x_1-h)-(x_1)}$$

Right-hand Derivatives

Progressive derivative or right-hand derivative of $f(x)$ at $x = x_1$ is given by

$$\text{RHD} = Rf'(x_1) = \lim_{h\to 0}\frac{f(x_1+h)-f(x_1)}{(x_1+h)-(x_1)}$$

Illustration 5. Prove that the function $f(x) = |x| + |x-1|$ is not differentiable at $x = 1$.

Solution: $f(x) = |x| + |x-1| = \begin{cases} -2x+1, & x<0 \\ 1, & 0 \le x < 1 \\ 2x-1, & 1 \le x \end{cases}$

$$\text{LHL} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1 = 1$$

$$\text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2x-1) = 1$$

and $\quad f(1) = 2\times 1 - 1 = 1$

$\because \quad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

$\therefore \quad f(x)$ is continuous at $x = 1$,

Now, $$Rf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{2(1+h)-1-1}{h} = 2$$

$$Lf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{1-1}{-h} = 0$$

$\because \quad Rf'(1) \ne Lf'(1)$

$\therefore \quad f(x)$ is not differentiable at $x = 1$.

RELATION BETWEEN CONTINUITY AND DIFFERENTIABILITY

If a function is differentiable at a point, then it is necessarily continuous at that point. But the converse is not necessarily true, i.e. every continuous function need not be differentiable.

PROPERTIES OF DIFFERENTIABLE FUNCTIONS

1. Every polynomial function is differentiable at each $x \in \mathbb{R}$.
2. The exponential function a^x, $a > 0$, is differentiable at each $x \in \mathbb{R}$.
3. Every constant function is differentiable at each $x \in \mathbb{R}$.
4. The logarithmic function is differentiable at each point in its domain.
5. Trigonometric and inverse trigonometric functions are differentiable in their respective domains.
6. The sum, difference, product and quotient of two differentiable functions are differentiable.
7. The composition of a differentiable function is a differentiable function.
8. Absolute functions are always continuous throughout but not differentiable at their critical points.

SOLVED PROBLEMS

1. The value of $\lim_{x\to 0^+}\frac{1}{3x}$ is

(a) $-\infty$ (b) -1
(c) 0 (d) $+\infty$

Ans. (d)

Solution: Expression is $\lim_{x\to 0^+}\frac{1}{3x}$.

We know that $\lim_{x\to 0^+}\frac{1}{3x} = \frac{1}{3\times 0} = \frac{1}{0} = +\infty$

2. The right-hand limit of the function sec x at $x = -(\pi/2)$ is

(a) $-\infty$ (b) -1
(c) 0 (d) ∞

Ans. (d)

Solution: Function $f(x) = \sec x$ and point $x = -(\pi/2)$. We know that right-hand limit of the function $f(x)$ at $x = -(\pi/2)$ is

$$\lim_{x\to[-(\pi/2)]^+} f(x) = \lim_{x\to[-(\pi/2)]} \sec x.$$