From (*iii*), the slope of the tangent to the curve (*i*) at $P(a, a) = -\frac{a}{a} = -1$ and from (*iv*), the slope of the tangent to the curve (*ii*) at $P(a, a) = -\frac{a}{a} = -1$.

Thus, the slope of tangent to curve (*i*) at P = slope of tangent to curve (*ii*) at P $\Rightarrow$ the given curves touch at $P(a, a)$.

Similarly, the given curves touch at the point $Q(-a, -a)$.

Hence, the given curves touch each other.

Example 18. *Show that the curves $x = y^2$ and $xy = k$ cut orthogonally if $8k^2 = 1$.*

Solution. The given curves are

$x = y^2$ ...(*i*) and $xy = k$ ...(*ii*)

Solving (*i*) and (*ii*) simultaneously for the points of intersection, we get

$$y^2 . y = k \Rightarrow y^3 = k \Rightarrow y = k^{1/3}.$$

From (*i*), $x = (k^{1/3})^2 \Rightarrow x = k^{2/3}$.

Thus, the two given curves intersect at the point $P(k^{2/3}, k^{1/3})$.

Differentiating (*i*) and (*ii*) w.r.t. x, we get $1 = 2y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$ ...(*iii*)

and $x . \frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ ...(*iv*)

From (*iii*), the slope of the tangent to curve (*i*) at $P = \frac{1}{2k^{1/3}}$.

From (*iv*), the slope of the tangent to curve (*ii*) at $P = -\frac{k^{1/3}}{k^{2/3}} = -\frac{1}{k^{1/3}}$.

Now, the given curves (*i*) and (*ii*) will cut orthogonally at P if

$$m_1 m_2 = -1 \Rightarrow \frac{1}{2k^{1/3}} . \left(-\frac{1}{k^{1/3}}\right) = -1$$

$$\Rightarrow \quad 2k^{2/3} = 1 \Rightarrow (2k^{2/3})^3 = 1^3 \Rightarrow 8k^2 = 1.$$

Hence, the given curves will cut orthogonally if $8k^2 = 1$.

EXERCISE 6.2

1. What is the slope of the tangent to the following curves :
   (*i*) $y = 3x^4 - 4x$ at $x = 4$?
   (*ii*) $x^2 + 3y + y^2 = 5$ at the point (1, 1)?
   (*iii*) $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3?
2. What is the slope of the normal to the following curves :
   (*i*) $y = x^3 - 5x^2 - x + 1$ at the point (1, –4)? (*ii*) $y = 2x^2 + 3e^x$ at $x = 0$?
3. Find the point on the curve $y = x^2 - 2x + 3$ at which the tangent is parallel to x-axis.
4. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$ at $x = 10$.
5. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel.

6. Find the points on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangents are

(a) parallel to x-axis (b) parallel to y-axis.

7. Find the points on the curve $y = x^3 - 3x^2 + 2x$ at which the tangent lines are parallel to the line $y - 2x + 3 = 0$.

8. Find the equations of the tangent and the normal to each of the following curves at the given point:

(i) $y = x^3$ at $(1, 1)$

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $(1, 3)$

(iii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $(0, 5)$

(iv) $y = x^2$ at $(0, 0)$

9. Find the equation of the normal to the curve $ay^2 = x^3$ at the point (am^2, am^3).

10. Find the equations of the tangent and the normal to the curve $x^{2/3} + y^{2/3} = 2$ at $(1, 1)$.

11. Find the equations of all lines having slope 2 and being tangents to the curve $y + \frac{2}{x-3} = 0$.

12. Find the equations of the normals to the curve $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.

13. Find the equation of the normal to the curve $y = x + \frac{1}{x}$, $x > 0$, perpendicular to the line $3x - 4y = 7$.

14. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $(-1, 4)$.

15. Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

16. Find the condition that the curves $2x = y^2$ and $2xy = k$ intersect orthogonally.

Answers

1. (i) 764 (ii) $-\frac{2}{5}$ (iii) 24

2. (i) $\frac{1}{8}$ (ii) $-\frac{1}{3}$

3. $(1, 2)$

4. $-\frac{1}{64}$

6. (i) $(0, 5), (0, -5)$ (ii) $(2, 0), (-2, 0)$

7. $(0, 0), (2, 0)$

8. (i) $3x - y - 2 = 0; x + 3y - 4 = 0$ (ii) $2x - y + 1 = 0; x + 2y - 7 = 0$
(iii) $10x + y - 5 = 0; x - 10y + 50 = 0$ (iv) $y = 0; x = 0$

9. $2x + 3my - am^2(2 + 3m^2) = 0$

10. $x + y - 2 = 0; x - y = 0$

11. $y - 2x + 2 = 0, y - 2x + 10 = 0$

12. $x + 3y + 8 = 0, x + 3y - 8 = 0$

13. $8x + 6y - 31 = 0$

14. $x - y + 3 = 0$

16. $k^2 = 8$

INCREASING AND DECREASING FUNCTIONS

In this section, we shall discuss monotonocity of functions. Monotonocity of a function is connected with the sign of its derivative. In determining the intervals of monotonocity of a function in its domain, we shall have to solve inequalities $f'(x) > 0, f'(x) \geq 0, f'(x) < 0$ and $f'(x) \leq 0$. Therefore, we first study a procedure of solving inequalities.