$$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{\frac{4}{3}} - 6\left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{-9}{4}$$

$$f(1) = 12(1)^{\frac{4}{3}} - 6(1)^{\frac{1}{3}} = 6$$

Hence, we conclude that absolute maximum value of f is 18 that occurs at $x = -1$ and absolute minimum value of f is $\frac{-9}{4}$ that occurs at $x = \frac{1}{8}$.

Example 29 An Apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier, placed at (3, 7), wants to shoot down the helicopter when it is nearest to him. Find the nearest distance.

Solution For each value of x, the helicopter's position is at point $(x, x^2 + 7)$. Therefore, the distance between the helicopter and the soldier placed at (3,7) is

$$\sqrt{(x-3)^2 + (x^2+7-7)^2}, \text{ i.e., } \sqrt{(x-3)^2 + x^4}.$$

Let $f(x) = (x-3)^2 + x^4$

or $f'(x) = 2(x-3) + 4x^3 = 2(x-1)(2x^2 + 2x + 3)$

Thus, $f'(x) = 0$ gives $x = 1$ or $2x^2 + 2x + 3 = 0$ for which there are no real roots. Also, there are no end points of the interval to be added to the set for which f' is zero, i.e., there is only one point, namely, $x = 1$. The value of f at this point is given by $f(1) = (1-3)^2 + (1)^4 = 5$. Thus, the distance between the solider and the helicopter is $\sqrt{f(1)} = \sqrt{5}$.

Note that $\sqrt{5}$ is either a maximum value or a minimum value. Since

$$\sqrt{f(0)} = \sqrt{(0-3)^2 + (0)^4} = 3 > \sqrt{5},$$

it follows that $\sqrt{5}$ is the minimum value of $\sqrt{f(x)}$. Hence, $\sqrt{5}$ is the minimum distance between the soldier and the helicopter.

EXERCISE 6.3

1. Find the maximum and minimum values, if any, of the following functions given by
   (i) $f(x) = (2x-1)^2 + 3$ (ii) $f(x) = 9x^2 + 12x + 2$
   (iii) $f(x) = -(x-1)^2 + 10$ (iv) $g(x) = x^3 + 1$

2. Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = |x + 2| - 1$ (ii) $g(x) = -|x + 1| + 3$

(iii) $h(x) = \sin(2x) + 5$ (iv) $f(x) = |\sin 4x + 3|$

(v) $h(x) = x + 1, x \in (-1, 1)$

3. Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(i) $f(x) = x^2$ (ii) $g(x) = x^3 - 3x$

(iii) $h(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$

(iv) $f(x) = \sin x - \cos x, \ 0 < x < 2\pi$

(v) $f(x) = x^3 - 6x^2 + 9x + 15$ (vi) $g(x) = \frac{x}{2} + \frac{2}{x}, \quad x > 0$

(vii) $g(x) = \frac{1}{x^2 + 2}$ (viii) $f(x) = x\sqrt{1-x}, \ 0 < x < 1$

4. Prove that the following functions do not have maxima or minima:

(i) $f(x) = e^x$ (ii) $g(x) = \log x$

(iii) $h(x) = x^3 + x^2 + x + 1$

5. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(i) $f(x) = x^3, x \in [-2, 2]$ (ii) $f(x) = \sin x + \cos x , x \in [0, \pi]$

(iii) $f(x) = 4x - \frac{1}{2}x^2, \ x \in \left[-2, \frac{9}{2}\right]$ (iv) $f(x) = (x-1)^2 + 3, \ x \in [-3, 1]$

6. Find the maximum profit that a company can make, if the profit function is given by

$$p(x) = 41 - 72x - 18x^2$$

7. Find both the maximum value and the minimum value of

$3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval [0, 3].

8. At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?

9. What is the maximum value of the function $\sin x + \cos x$?

10. Find the maximum value of $2x^3 - 24x + 107$ in the interval [1, 3]. Find the maximum value of the same function in [–3, –1].

11. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 2]. Find the value of a.

12. Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.

13. Find two numbers whose sum is 24 and whose product is as large as possible.

14. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

15. Find two positive numbers x and y such that their sum is 35 and the product $x^2 y^5$ is a maximum.

16. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

17. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.

18. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?

19. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

20. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

21. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?

22. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

24. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

25. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

26. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

Choose the correct answer in Questions 27 and 29.

27. The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is

(A) $(2\sqrt{2},4)$ (B) $(2\sqrt{2},0)$ (C) $(0, 0)$ (D) $(2, 2)$

28. For all real values of x, the minimum value of $\dfrac{1-x+x^2}{1+x+x^2}$ is

(A) 0 (B) 1 (C) 3 (D) $\dfrac{1}{3}$

29. The maximum value of $[x(x-1)+1]^{\frac{1}{3}}$, $0 \le x \le 1$ is

(A) $\left(\dfrac{1}{3}\right)^{\frac{1}{3}}$ (B) $\dfrac{1}{2}$ (C) 1 (D) 0

Miscellaneous Examples

Example 30 A car starts from a point P at time $t = 0$ seconds and stops at point Q. The distance x, in metres, covered by it, in t seconds is given by

$$x = t^2\left(2-\frac{t}{3}\right)$$

Find the time taken by it to reach Q and also find distance between P and Q.

Solution Let v be the velocity of the car at t seconds.

Now $$x = t^2\left(2-\frac{t}{3}\right)$$

Therefore $$v = \frac{dx}{dt} = 4t - t^2 = t(4-t)$$

Thus, $v = 0$ gives $t = 0$ and/or $t = 4$.

Now $v = 0$ at P as well as at Q and at P, $t = 0$. So, at Q, $t = 4$. Thus, the car will reach the point Q after 4 seconds. Also the distance travelled in 4 seconds is given by

$$x]_{t=4} = 4^2\left(2-\frac{4}{3}\right) = 16\left(\frac{2}{3}\right) = \frac{32}{3}\text{ m}$$