10

LEJEUNE DIRICHLET

Functions

DEFINITION OF FUNCTION

Function: Let X and Y be two nonempty sets. A function f, defined from X to Y, is a rule or a collection of rules, which associates to each element x in X a unique element y in Y.

Symbolically, we write it as $f: X \rightarrow Y$, which is read as "function defined from the set X to the set Y".

(i) The unique element y of Y is called the value of f at x (the image of x under f). It is written as $f(x)$. Thus, $y = f(x)$.

(ii) The element x of X is called pre-image (or inverse image) of y.

(iii) The set X is called the domain of f.

(iv) The set Y is called the co-domain of f.

(v) The set consisting of all images of the elements of X under f is called the range of f. This is denoted by $f(x)$.
Thus, the range of $f = f(x) = \{f(x)$: for all $x \in X\}$.
This is a subset of Y, which may or may not be equal to Y.

Note A function is also termed mapping or correspondence or transformation.

Examples:

(i) Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8, 10\}$.
Here the rule, i.e. f, which associates to each element x in A, the element $2x$ in B is a function from A to B. The rule written as $f(x) = 2x$ is depicted in Fig. 10.1.

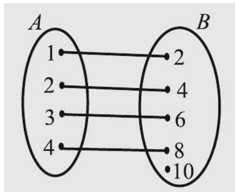

Fig. 10.1

(ii) Let $A = \{a, b, c, d, e\}$ and $B = \{p, q, r\}$.
Here the rule, which is depicted in Fig. 10.2, is not a function from A to B because the elements a and c in A have been associated with two elements p and q each of B.

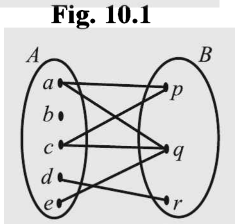

Fig. 10.2

WORKING RULES FOR FINDING DOMAIN OF A FUNCTION

1. Expression under even root (i.e. square root, fourth root) is greater than or equal to zero.
2. Denominator should not be zero.
3. If the domains of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively, then the domain of $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ *is* $D_1 \cap D_2$, while domain of $\dfrac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{x : g(x) = 0\}$.

Illustration 1. Find the domain of $f(x) = \sqrt{\cos(\sin x)}$.

Solution: $f(x) = \sqrt{\cos(\sin x)}$ is defined if

$$\cos(\sin x) \geq 0 \qquad \text{(i)}$$

As we know $-1 \leq \sin x \leq 1$ for all x.

$$\cos\theta \geq 0$$

[Here, $\theta = \sin x$ lies in the third and fourth quadrants.]

i.e. $\cos(\sin x) \geq 0$ for all x, i.e. $x \in \mathbb{R}$

Thus, the domain of $f(x) \in \mathbb{R}$.

WORKING RULES FOR FINDING RANGE OF A FUNCTION

Step 1: Find the domain for the function $y = f(x)$.

Step 2: Change the function $y = f(x)$ as $x = f(y)$.

Step 3: Solve $x = \phi(y)$.

Step 4: Find the values of y for x in the domain of f.

Step 5: The set of values of y in step 4 is the range of $f(x)$.

Illustration 2. Find the range of $f(x) = \sin^2 x - \sin x + 1$.

Solution: $f(x) = \sin^2 x - \sin x + 1 = \left(\sin x - \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$

Now, $-1 \leq \sin x \leq 1 \quad \Rightarrow \quad -\dfrac{3}{2} \leq \sin x - \dfrac{1}{2} \leq \dfrac{1}{2}$

$$\Rightarrow \quad 0 \leq \left(\sin x - \frac{1}{2}\right)^2 \leq \frac{9}{4}$$

$$\Rightarrow \quad \frac{3}{4} \leq \left(\sin x - \frac{1}{2}\right)^2 + \frac{3}{4} \leq 3$$

Hence, the range is $\left[\dfrac{3}{4}, 3\right]$.

TYPES OF FUNCTIONS

1. Constant Function

A function which does not change as its parameters vary, i.e. the function whose rate of change is zero, is called a *constant function*. In short, we can say that a constant function is a function that always gives or returns the same value.

Domain: $x \in (-\infty, \infty)$, i.e. $x \in \mathbb{R}$

Range: $y \in \{c\}$

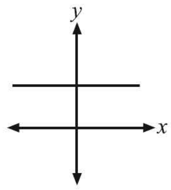

Fig. 10.3

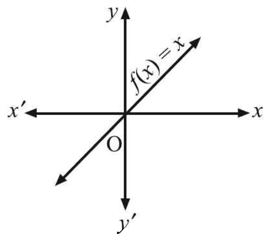

Fig. 10.4

2. Identity Function

An identity function is a function of the form $y = f(x) = x$, which is a straight line passing through the origin and having slope unity.

Domain: $x \in (-\infty, \infty)$, i.e. $x \in \mathbb{R}$

Range: $y \in (-\infty, \infty)$, i.e. $y \in \mathbb{R}$

3. Polynomial Function

A polynomial function is a function of the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$$

Here the value of n must be a non-negative integer, i.e. it must be a whole number and $a_a, a_1, a_2, \ldots, a_n \in \mathbb{R}$. The degree of the polynomial is the highest value for n, where $a_n \neq 0$, i.e. n.

Domain: The entire number line, i.e. $x \in \mathbb{R}$

Range: Varies from function to function.

4. Rational Function

If $h(x)$ and $g(x)$ are polynomial functions, $g(x) \neq 0$, then the function $f(x) = \dfrac{h(x)}{g(x)}$ is known as a *rational function*.

Domain: $x \in \mathbb{R} - \{x : g(x) = 0\}$

Range: Varies from function to function.

5. Irrational Function

An *irrational function* is a function of the form

$$y = \{f(x)\}^{p/q}$$

where p and q are integers, have no common factor and also $q \neq 0$, $q > 0$. For example,

$$y = f(x) = \frac{4x^{5/2} - 9x^{3/2}}{x^{1/2} - 7}$$

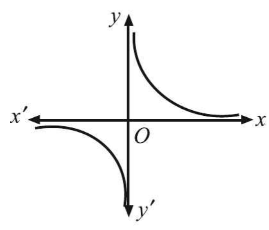

Fig. 10.5

6. Reciprocal Function

A *reciprocal function* is the inverse of an identity function, such that $y = f(x) = \dfrac{1}{x}$, which is an equation of a rectangular hyperbola whose asymptotes (here x-axis and y-axis) intersect at right angles.

Domain: $x \in \mathbb{R} - \{0\}$
Range: $y \in \mathbb{R} - \{0\}$

7. Exponential Function

A function of the form $f(x) = a^x$ is an *exponential function*. It is the inverse of a logarithmic function. The value of the function depends upon the value of a.

For $0 < a < 1$, the function decreases, and for $a > 1$, the function increases.
Domain: $x \in \mathbb{R}$
Range: $y \in \mathbb{R}^+$

8. Logarithmic Function

A function of the form $f(x) = \log_a x$ is a *logarithmic function*. The value of the function depends upon the value of a. For $0 < a < 1$, the function decreases, and for $a > 1$, the function increases.
Domain: $x > 0$, i.e. $x \in (0, \infty)$
Range: $y \in \mathbb{R}$

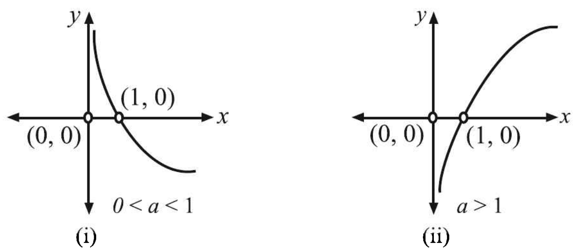

Fig. 10.6

8.1 *Properties of a logarithmic function*

1. $\log_e(xy) = \log_e x + \log_e y \quad x, y > 0$
2. $\log_e\left(\frac{x}{y}\right) = \log_e x - \log_e y \quad x, y > 0$
3. $\log_e x^a = a\log_e x \quad x > 0$
4. $\log_b a = \frac{\log_c a}{\log_c b}$, where c is any constant such that $c \in (0, \infty) - \{1\}$. Also, $a, b > 0$

   (i) $\log_a a = 1$ (ii) $\log_b a = \frac{1}{\log_a b}$
5. $a^{\log_a b} = b$
6. If $\log_b a = c$, then $a = b^c$, where $a > 0$ and $b \in (0, \infty) - \{1\}$
7. $a^{\log_c b} = b^{\log_c a}$
8. If $\log_a x > \log_a y$, and if $a > 1$ then $x > y$
   i.e. for $a > 1$, the inequality remains the same, else if $0 < a < 1$ then $x < y$, and for $0 < a < 1$, the inequality gets reversed.

Illustration 3. Find the domain of the function

$$f(x) = \left[\log_{10}\left(\frac{5x - x^2}{4}\right)\right]^{1/2}.$$

Solution: $f(x) = \left[\log_{10}\left(\frac{5x - x^2}{4}\right)\right]^{1/2}$ (i)

From (i), clearly $f(x)$ is defined for those values of x for which

$$\log_{10}\left[\frac{5x - x^2}{4}\right] \geq 0.$$

$$\Rightarrow \left(\frac{5x - x^2}{4}\right) \geq 10^0 \Rightarrow \left(\frac{5x - x^2}{4}\right) \geq 1$$

$$\Rightarrow x^2 - 5x + 4 \leq 0 \Rightarrow (x - 1)(x - 4) \leq 0$$

Hence, the domain of the function is [1, 4].

9. Trigonometric Function

A function of trigonometric ratios is known as *trigonometric function*.

Function	Domain	Range
$\sin x$	$\mathbb{R}$	$[-1, 1]$
$\cos x$	$\mathbb{R}$	$[-1, 1]$
$\tan x$	$\mathbb{R} - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$	$\mathbb{R}$
$\cot x$	$\mathbb{R} - \{n\pi : n \in I\}$	$\mathbb{R}$
$\sec x$	$\mathbb{R} - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$	$(-\infty, -1] \cup [1, \infty)$
$\operatorname{cosec} x$	$\mathbb{R} - \{n\pi : n \in I\}$	$(-\infty, -1] \cup [1, \infty)$

10. Inverse Trigonometric Function

A function involving inverse trigonometric ratios is known as *inverse trigonometric function*.

Function	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\operatorname{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\cot^{-1} x$	$\mathbb{R}$	$(0, \pi)$

11. Modulus Function

A function $y = f(x) = |x|$ is known as *modulus function*.

$$y = f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Domain: $f(x) = x \in \mathbb{R}$

Range: $f(x) = [0, \infty)$

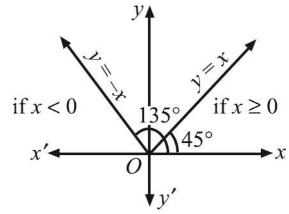

Fig. 10.7

11.1 *Properties of a modulus function*

1. For any real number x, $\sqrt{x^2} = |x|$.
2. If a, b are positive real numbers, then
   (i) $x^2 \leq a^2 \Leftrightarrow |x| \leq a \Leftrightarrow -a \leq x \leq a$
   (ii) $x^2 \geq a^2 \Leftrightarrow |x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$
   (iii) $a^2 \leq x^2 \leq b^2 \Leftrightarrow a \leq |x| \leq b \Leftrightarrow x \in [-b, -a] \cup [a, b]$
3. $|x + y| \geq ||x| - |y||$
4. $|x \pm y| \leq |x| + |y|$

12. Greatest Integer Function

The greatest integer function is denoted as $y = f(x) = [x]$, where $[x]$ means the greatest integer n, which is less than or equal to x, i.e. $n \leq x$.

$[x]$ = [Integer + Proper positive fraction] = Integer

For example,

$[4.2] = [4 + 0.2] = 4$

$[-4.2] = [-5 + 0.8] = -5$

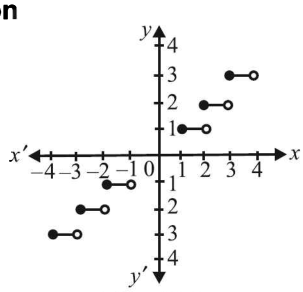

Fig. 10.8

12.1 *Properties of a greatest integer function*

1. If $[x] = x$, then x must belong to the set of integers
2. $[x + I] = [x] + [I] = [x] + I$
3. $[x + y] \geq [x] + [y]$
4. If $[f(x)] \geq I$, then $f(x) \geq I$
   If $[f(x)] \leq I$, then $f(x) < I + 1$
5. $[-x] = \begin{cases} -[x] & x \in I \quad \text{i.e. } [x] + [-x] = 0 \\ -[x] - 1 & x \notin I \quad \text{i.e. } [x] + [-x] = -1 \end{cases}$

13. Smallest Integer Function (Ceiling Function)

For any real number x, we use the symbol $\lceil x \rceil$ to denote the smallest integer greater than or equal to x.

The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lceil x \rceil$ for all $x \in \mathbb{R}$ is called the *smallest integer function* or the *ceiling function*. It is also a step function.

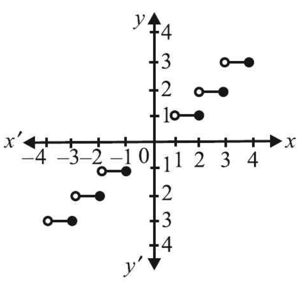

Fig. 10.9

Domain: $x \in \mathbb{R}$

Range: $y \in I$

14. Fractional Part Function

$y = f(x) = \{x\}$, by definition. Mathematically,

$$\{x\} = x - [x]$$

It always returns the fractional part of the variable and is always positive.

Domain: $x \in \mathbb{R}$

Range: $y \in [0, 1)$

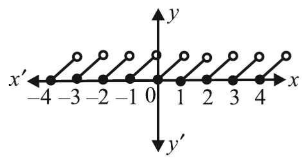

Fig. 10.10

14.1 *Properties of a fractional part function*

1. If $0 \leq x < 1$ then $\{x\} = x$
2. If $x \in I$ then $\{x\} = 0$
3. If $x \notin I$ and $x > 0$ then $-\{x\} = 1 - \{x\}$

15. Signum Function

This function is defined by

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

Thus, we have

$$f(x) = \begin{cases} 1, & \text{when } x > 0 \\ 0, & \text{when } x = 0 \\ -1, & \text{when } x < 0 \end{cases}$$

Domain: $x \in \mathbb{R}$

Range: $y \in \{-1, 0, 1\}$

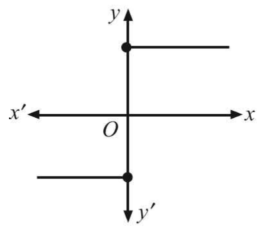

Fig. 10.11

EVEN ODD FUNCTIONS

1. **Even function:** A function f is said to be an even function of x if $f(-x) = f(x)$. For example, $f(x) = x^2 + 4$, $f(x) = 3\cos x - 5$, etc. are all even functions of x.
2. **Odd function:** A function f is said to be an odd function of x if $f(-x) = -f(x)$. For example, $f(x) = x^3 + 7x$, $f(x) = 7\sin^3 x + \tan x$, etc. are all odd functions of x.
3. **Extension of a function:** If a function $f(x)$ is defined in $[0, a]$, then $0 \leq x \leq a \Rightarrow -a \leq -x \leq 0$
   $\Rightarrow x \in [-a, 0]$.
   We define $f(x)$ in $[-a, 0]$ such that $f(x) = f(-x)$.
   If g is the even extension, then

$$g(x) = \begin{cases} f(x), & x \in [0, a] \\ f(-x), & x \in [-a, 0] \end{cases}$$

4. **Odd extension:** If $f(x)$ is defined in $[0, a]$, then
$0 \le x \le a \quad \Rightarrow \quad -a \le -x \le 0$
$\Rightarrow \; -x \in [-a, 0]$.
We define $f(x)$ in $[-a, 0]$ such that $f(x) = -f(-x)$.
If g is the odd extension, then
$$g(x) = \begin{cases} f(x); & x \in [0, a] \\ -f(-x); & x \in [-a, 0] \end{cases}$$

PERIODIC FUNCTION

A function f is said to be periodic with period α if
$$f(x + \alpha) = f(x) \text{ for all } x,$$
where α is the least positive real number. For example, sin x, cos x, sec x and cosec x are all periodic functions with period 2π and tan x and cot x are periodic functions with period π.

Standard results on some periodic functions

S. No.	Function	Period
1.	$\sin^n x$, $\cos^n x$, $\text{cosec}^n x$	π (if n is an even number) 2π (if n is an odd number)
2.	$\tan^n x$, $\cot^n x$	π
3.	$\|\sin x\|$, $\|\cos x\|$, $\|\tan x\|$, $\|\sec x\|$ $\|\text{cosec } x\|$, $\|\cot x\|$	π
4.	$x - [x]$	1
5.	$\sin^{-1}(\sin x)$, $\cos^{-1}(\cos x)$, $\text{cosec}^{-1}(\text{cosec } x)$, $\sec^{-1}(\sec x)$	2π
6.	$\tan^{-1}(\tan x)$, $\cot^{-1}(\cot x)$	π

PROPERTIES OF A PERIODIC FUNCTION

1. If $f(x)$ is periodic with period p, then $af(x) + b$, where $a, b \in \mathbb{R}$ $(a \ne 0)$, is also periodic with period p.
2. If $f(x)$ is periodic with period p, then $f(ax) + b$, where $a, b \in \mathbb{R}$ $(a \ne 0)$, is also periodic with period $\dfrac{p}{|a|}$.
3. Suppose $f(x)$ has period $p = \dfrac{m}{n}$ $(m, n \in N$ and co-prime$)$ and $g(x)$ has period $q = \dfrac{r}{s}$ $(r, n \in N$ and co-prime$)$ and let t be the LCM of p and q, i.e. $\dfrac{\text{LCM of } (m, r)}{\text{HCF of } (r, s)}$.
Then t shall be the period of $f + g$ provided there does not exist a positive number k $(< t)$ for which $f(x + k) + g(x + k) = f(x) + g(x)$, else k will be the period.
The same rule is applicable for any other algebraic combination of $f(x)$ and $g(x)$.
The LCM of p and q always exists if p/q is a rational quantity. If p/q is irrational, then the algebraic combination of f and g is non-periodic.

Illustration 4. Find the periods (if periodic) of $f(x) = \tan \pi/2$ where $[\cdot]$ denotes the greatest integer function.

Solution: $f(x) = \tan \dfrac{\pi}{2}[x], \tan \dfrac{\pi}{2}[x + T]$
$$= \tan \frac{\pi}{2}[x] \;\Rightarrow\; \tan \frac{\pi}{2}[x + T]$$
$$= n\pi + \frac{\pi}{2}[x] \;\Rightarrow\; \text{Period} = 2.$$

COMPOSITION OF FUNCTION

Let A, B and C be three nonempty sets. Let $f : A \to B$ and $g : B \to C$ be two mappings (or functions), then $g \circ f : A \to C$. This function is called the product or composite of f and g given by the function
$$g \circ f(x) = g[f(x)], \text{ for all } x \in A$$

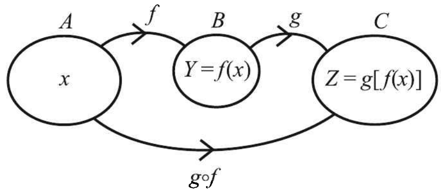

Fig. 10.12

$g \circ f$ exists if the range of f is a subset of the domain of g. Similarly, $f \circ g$ exists if the range of g is a subset of the domain of f.

Note

- The $g \circ f$ is defined only if for all $x \in A$, $f(x)$ is an element of the domain of g so that we can take its g-image.
- The range of f must be a subset of the domain of g in $g \circ f$.
- (i) $(f \circ g)(x) = f[g(x)]$
  (ii) $(f \circ f)(x) = f[f(x)]$
  (iii) $(g \circ g)(x) = g[g(x)]$
  (iv) $(f \pm g)(x) = f(x) \pm g(x)$
  (v) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$; $g(x) \ne 0$

Properties of a Composite Function

1. The composition of a function is not commutative, i.e. $f \circ g \ne g \circ f$.
2. The composition of a function is associative, i.e. $f \circ (g \circ h) = (f \circ g) \circ h$.

3. The composition of any function with the identity function is the function itself, i.e. if $f: A \to B$, then $f \circ I_A = I_B \circ f = f$
4. Let $f: A \to B$, $g: B \to C$ be two functions, then
   (a) f and g are injective $\Rightarrow g \circ f$ is injective
   (b) f and g are surjective $\Rightarrow g \circ f$ is surjective
   (c) f and g are bijective $\Rightarrow g \circ f$ is bijective
5. An injective mapping from a finite set to itself is bijective.

SOLVED PROBLEMS

1. Let $f(x) = \dfrac{9^x}{9^x + 3}$. Then $f(x) + f(1-x)$ is

(a) 2 (b) 1
(c) −1 (d) none of these

Ans. (b)

Solution: $f(x) = \dfrac{9^x}{9^x + 3}$ (i)

and $f(1-x) = \dfrac{9^{1-x}}{9^{1-x} + 3}$

$$\Rightarrow \quad f(1-x) = \frac{\frac{9}{9^x}}{\frac{9}{9^x} + 3} = \frac{9}{9 + 3.9^x}$$

$$\Rightarrow \quad f(1-x) = \frac{3}{(3 + 9^x)} \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$f(x) + f(1-x) = \frac{9^x}{9^x + 3} + \frac{3}{3 + 9^x} = 1$$

$$\Rightarrow \quad f(x) + f(1-x) = 1$$

2. The expression $\left[x + \sqrt{x^2 - 1}\right]^5 + \left[x - \sqrt{x^2 - 1}\right]^5$ is a polynomial of degree

(a) 5 (b) 6
(c) 10 (d) 20

Ans. (a)

Solution: $\left[x + \sqrt{x^2 - 1}\right]^5 + \left[x - \sqrt{x^2 - 1}\right]^5$

$$= x^5 + {}^5C_1 x^4 \sqrt{x^2-1} + {}^5C_2 x^3 (x^2 - 1)$$
$$+ {}^5C_3 x^2 (x^2-1)^{3/2} + {}^5C_4 x (x^2-1)^2 + {}^5C_5 (x^2-1)^{5/2}$$
$$+ x^5 - {}^5C_1 x^4 \sqrt{x^2-1} + {}^5C_2 x^3 (x^2-1)$$
$$- {}^5C_3 x^2 (x^2-1)^{3/2} + {}^5C_4 x (x^2-1) - {}^5C_5 (x^2-1)^{5/2}$$
$$= 2x^5 + 2\,{}^5C_2 x^3 (x^2-1) + 2\,{}^5C_4 x (x^2-1)^2,$$

which is a polynomial of degree 5.

3. Let $f(x) = 4\cos\sqrt{x^2 - \dfrac{\pi^2}{9}}$. Then

(a) the domain of f is $\left[\dfrac{\pi}{3}, +\infty\right)$
(b) the range of f is $[-1, 1]$
(c) the domain of $f \left(-\infty, -\dfrac{\pi}{3}\right] \cup \left[\dfrac{\pi}{3}, +\infty\right)$
(d) none of these

Ans. (c)

Solution: Since $f(x) = 4\cos\sqrt{x^2 - \dfrac{\pi^2}{9}}$

By definition, the domain of f can be defined if

$$x^2 - \frac{\pi^2}{9} \geq 0 \quad \Rightarrow \quad x^2 \geq \left(\frac{\pi}{3}\right)^2$$

$$\Rightarrow \quad |x| \geq \frac{\pi}{3}$$

either $x \leq -\dfrac{\pi}{3}$ or $x \geq \dfrac{\pi}{3}$

i.e. $D_f = x \in \left(-\infty, -\dfrac{\pi}{3}\right] \cup \left[\dfrac{\pi}{3}, \infty\right)$.

4. The domain of the function $f(x) = \dfrac{1}{\sqrt{x^2 - 3x + 2}}$ is

(a) $(-\infty, 1)$ (b) $(-\infty, 1) \cap (2, \infty)$
(c) $(-\infty, 1] \cup [2, \infty)$ (d) $(2, \infty)$

Ans. (b)

Solution: For $f(x)$ to be defined, we must have

$$x^2 - 3x + 2 = (x-1)(x-2) > 0$$

$\Rightarrow$ $x < 1$ or > 2

Domain of $f = (-\infty, 1) \cup (2, \infty)$

5. The function $f(x) = \sin[\log(x + \sqrt{x^2 + 1})]$ is

(a) even (b) odd
(c) neither even nor odd (d) periodic

Ans. (b)

Solution: $f(x) = \sin[\log(x + \sqrt{1 + x^2})]$

$$\Rightarrow \quad f(-x) = \sin[\log(-x + \sqrt{1 + x^2})]$$

$$\Rightarrow \quad f(-x) = \sin\log\left[(\sqrt{1+x^2} - x)\frac{(\sqrt{1+x^2} + x)}{(\sqrt{1+x^2} + x)}\right]$$

$$\Rightarrow \quad f(-x) = \sin\log\left[\frac{1}{(x + \sqrt{1+x^2})}\right]$$