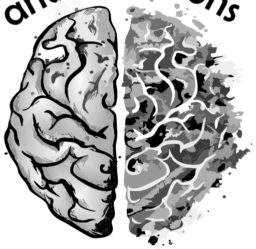
Relations and Functions



Types of Relations

Reflexive relation: A relation $R : A \to A$ is reflexive if $aRa \ \forall \ a \in A$

Symmetric relation: A relation $R : A \to A$ is symmetric if $aRb \Rightarrow bRa \ \forall \ a, b \in A$

Transitive relation: A relation $R : A \to A$ is transitive if aRb and $bRc \Rightarrow aRc \ \forall \ a,b,c \in A$.

Reflexive + Symmetric + Transitive →

Equivalence relation

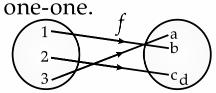
(If a relation has reflexive, symmetric and transitive relations) e.g., Let T = the set of all triangles in a plane and $R : T \to T$ defined by $R = \{(T_1, T_2)\} : T_1$ is congruent to $T_2\}$.Then, R is equivalence.

Types of Functions

One-one (injective)

$f : X \to Y$ is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ $\forall \ x_1, x_2 \in X$. Otherwise, f is many- one.

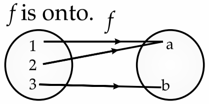
f is one-one.



(Surjective) Onto

$f : X \to Y$ is onto if every $y \in Y, \exists \ x \in X$ such that $f(x) = y$, Then f is surjective

f is onto.



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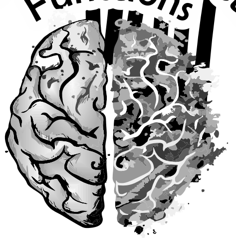
Inverse Trigonometric Functions



Domain and range of inverse trigonometric functions

(i) $y = \sin^{-1}x$. Domain = $[-1,1]$, Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $y = \cos^{-1}x$. Domain = $[-1,1]$ Range = $[0, \pi]$

(iii) $y = \text{cosec}^{-1}x$. Domain = $\mathbb{R}-(-1,1)$, Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(iv) $y = \sec^{-1}x$. Domain = $\mathbb{R}-(-1,1)$, Range = $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

(v) $y = \tan^{-1}x$. Domain = $\mathbb{R}$, Range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vi) $y = \cot^{-1}x$. Domain = $\mathbb{R}$, Range = $(0, \pi)$.

$\sin^{-1}x \neq (\sin x)^{-1} = \frac{1}{\sin x}$

Trigonometric functions

(i) $\sin : \mathbb{R} \to [-1,1]$

(ii) $\cos : \mathbb{R} \to [-1,1]$

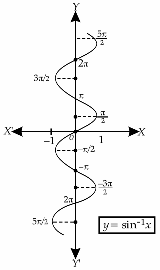
(iii) $\tan : \mathbb{R} - \left\{x : x = (2n+1)\frac{\pi}{2}, n \in Z\right\} \to \mathbb{R}$

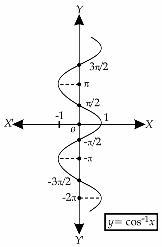
(iv) $\cot : \mathbb{R} - \{x : x = n\pi, n \in Z\} \to \mathbb{R}$

(v) $\sec : \mathbb{R} - \left\{x : x = (2n+1)\frac{\pi}{2}, n \in Z\right\} \to \mathbb{R} - (-1,1)$

(vi) $\text{cosec} : \mathbb{R} - \{x : x = n\pi, n \in Z\} \to \mathbb{R} - (-1,1)$

Graphs of trigonometric functions





Some important relations

(i) $y = \sin^{-1}x \Rightarrow x = \sin y$

(ii) $x = \sin y \Rightarrow y = \sin^{-1}x$

(iii) $\sin(\sin^{-1}x) = x, -1 \le x \le 1$

(iv) $\sin^{-1}(\sin x) = x, \frac{-\pi}{2} \le x \le \frac{\pi}{2}$

(v) $\sin^{-1}\frac{1}{x} = \text{cosec}^{-1}x$

(vi) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(vii) $\cos^{-1}\frac{1}{x} = \sec^{-1}x$

(viii) $\cot^{-1}(-x) = \pi - \cot^{-1}x$

(ix) $\tan^{-1}\frac{1}{x} = \cot^{-1}x, x > 0$

(x) $\sec^{-1}(-x) = \pi - \sec^{-1}x$

(xi) $\sin^{-1}(-x) = -\sin^{-1}x$

(xii) $\tan^{-1}(-x) = -\tan^{-1}x$

(xiii) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, -1 \le x \le 1$

(xiv) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(xv) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}, |x| \ge 1$

(xvi) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}, xy < 1$

(xvii) $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}, -1<x<1$

(xviii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}, xy > 1$

(xix) $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2}$

Principal value branch and principal value

The range of an inverse trigonometric function is the principal value branch and those values which lies in the principal value branch is called the principal value of that inverse trigonometric function.

If $x > = 0$	If $x > 0$
$0 \le \sin^{-1}x \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \sin^{-1}x < 0$
$0 \le \cos^{-1}x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1}x \le \pi$
$0 \le \tan^{-1}x \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \tan^{-1}x < 0$
$0 \le \cot^{-1}x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cot^{-1}x < \pi$
$0 \le \sec^{-1}x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \sec^{-1}x \le \pi$
$0 \le \text{cosec}^{-1}x \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \text{cosec}^{-1}x < 0$

How to understand Mind Map?

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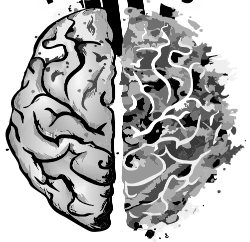
Matrices



Definition and its types

A matrix of order $m \times n$ is an ordered rectangular array of numbers or functions having 'm' rows and 'n' columns. The matrix $A=[a_{ij}]_{m\times n}$, $1 \le i \le m$, $1 \le j \le n$; $i, j \in N$ is given by

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ a_{m_1} & a_{m_2} & \dots a_{mn} \end{bmatrix}_{m\times n}$$

- **Column matrix :** It is of the form $\left[a_{ij}\right]_{m\times 1}$
- **Row matrix :** It is of the form $\left[a_{ij}\right]_{1\times n}$
- **Square matrix :** Here, $m = n$ (no. of rows = no. of columns)
- **Diagonal matrix :** All non-diagonal entries are zero i.e. $a_{ij} = 0 \forall i \neq j$
- **Scalar matrix :** $a_{ij} = 0, i \neq j$ and $a_{ij} = k$ (Scalar), $i = j$, for some constant k.
- **Identity matrix :** $a_{ij} = 0, i \neq j$ and $a_{ij} = 1, i = j$
- **Zero matrix :** All elements are zero.

Equality of two matrices

$A=\left[a_{ij}\right]=\left[b_{ij}\right]=B$ if, A and B are of same order and $a_{ij}=b_{ij} \forall$ i and j; $i,j \in N$

Transpose of a Matrix

If $A=[a_{ij}]_{m\times n}$, then its transpose $A' = (A^T) = [a_{ji}]_{n\times m}$ i.e. if $A=\begin{pmatrix} 2 & 1 \end{pmatrix}$ then $A^T = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Also, $(A')' = A$, $(kA)' = kA'$, $(A+B)' = A'+B'$, $(AB)' = B'A'$.

- A is symmetric matrix if $A=A'$ i.e. $A'=A$.
- A is skew – symmetric if $A=-A'$ i.e. $A'=-A$.
- A is any square matrix, then–

$$A=\frac{1}{2}\left\{\underset{\text{S.M.}}{(A+A')}+\underset{\text{Skew S.M.}}{(A-A')}\right\} = \text{sum of a symmetric and a skew-symmetric matrix.}$$

For example if $A=\begin{pmatrix} 2 & 8 \\ 6 & 4 \end{pmatrix}$, then $A=\frac{1}{2}\left\{\begin{pmatrix} 4 & 14 \\ 14 & 8 \end{pmatrix}+\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\right\}$.

Properties for applying the operations

$R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

$R_i \to kR_i$ or $C_i \to kC_i$

$R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

Operations on matrices

Multiplication

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{jk}]_{n\times p}$, then $AB = C = [C_{ik}]_{m\times p}$, $[C_{ik}] = \sum_{j=1}^{n} a_{ij} b_{jk}$.

Also, $A(BC) = (AB)C$, $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$, but $AB \neq BA$ (always).

Addition

If A, B are two matrices of same order, then $A+B=[a_{ij}+b_{ij}]$. The addition of A and B follows: $A+B=B+A$, $(A+B)+C = A+(B+C)$, $-A=(-1)A$, $k(A+B) = kA + kB$, k is scalar and $(k+I)A = kA + IA$, k and I are constants.

- If $A=\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$, $B=\begin{bmatrix} -3 & 2 \\ -4 & 5 \end{bmatrix}$ then $A+B=\begin{bmatrix} -1 & 5 \\ -2 & 9 \end{bmatrix}$
- If $A = [2\ 3]_{1\times 2}$, $B=\begin{bmatrix} 4 \\ 5 \end{bmatrix}_{2\times 1}$, then $AB = [2\times4+3\times5] = [23]_{1\times 1}$

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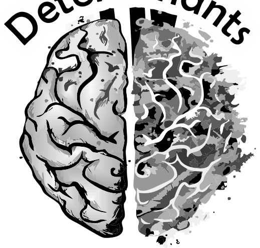
Determinants



Applications of determinants & matrices

- If $a_1x+b_1y+c_1z=d_1, a_2x+b_2y+c_2z=d_2, a_3x+b_3y+c_3z=d_3$ then we can write $AX=B$,

  where $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$
- Unique solution of $AX=B$ is $X=A^{-1}B$, $|A|\neq 0$.
- $AX=B$ is consistent or inconsistent according as the solution exists or not.
- For a square matrix A in $AX=B$, if
  - (i) $|A|\neq 0$ then there exists unique solution.
  - (ii) $|A|=0$ and (adj. A) $B\neq 0$, then no solution.
  - (iii) if $|A|=0$ and (adj.A).$B=0$ then system may or may not be consistent.

Area of a triangle

If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of triangle, Area of $\Delta=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

e.g., if (1, 2), (3, 4) and (–2, 5) are the vertices, then area of the triangle is

$$\Delta=\frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ -2 & 5 & 1 \end{vmatrix}=\frac{1}{2}\left|1(4-5)-2(3+2)+1(15+8)\right|=6 \text{ sq units.}$$

we take positive value of the determinant because area is considered positive.

Minors and cofactors of a matrix

Minor of an element a_{ij} in a determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and is denoted by M_{ij}. If M_{ij} is the minor of a_{ij} and cofactor of a_{ij} is A_{ij} given by $A_{ij}=(-1)^{i+j}M_{ij}$.

- If $A_{3\times3}$ is a matrix, then $|A|=a_{11}.A_{11}+a_{12}.A_{12}+a_{13}.A_{13}$.
- If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For e.g., $a_{11}A_{21}+a_{12}A_{22}+a_{13}A_{33}=0$.

e.g., if $A=\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, then $M_{11}=4$ and $A_{11}=(-1)^{1+1}4=4$.

$M_{12}=-3$ and $A_{12}=(-1)^{1+2}(-3)=4$.

Determinant of a square matrix 'A', |A| is given by

(i) if $A=[a_{11}]_{1\times1}$, then $|A|=a_{11}$

(ii) if $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}$ then $|A|=a_{11}a_{22}-a_{12}a_{21}$

(iii) if $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$, then $|A|=a_{11}(a_{22}.a_{33}-a_{23}.a_{32}) - a_{12}(a_{21}.a_{33}-a_{23}.a_{31}) + a_{13}(a_{21}a_{32}-a_{22}a_{31})$

e.g., If $A=\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$, then $|A|=2\times4-3\times2=2$

Adjoint and inverse of a matrix

If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $Adj\ A=\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$,

where A_{ij} is the cofactor of a_{ij}.

- A(adj.A)=(adj.A).$A=|A|I$, A – square matrix of order 'n'
- If $|A|=0$, then A is singular. Otherwise, A is non-singular.
- If $AB=BA=I$, where B is a square matrix, then B is called the inverse of A, $A^{-1}=B$ or $B^{-1}=A$, $(A^{-1})^{-1}=A$.

Inverse of a square matrix exists if A is non-singular i.e. $|A|\neq 0$, and is given by

$$A^{-1}=\frac{1}{|A|}(\text{adj.}A)$$

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Continuity and Differentiability

Continuous Function

Suppose f is a real function on a subset of the real numbers and let 'c' be a point in the domain of f.

Then f is continuous at c if $\lim_{x\to c} f(x) = f(c)$

A real function f is said to be continuous if it is continuous at every point in the domain of f.

eg: The function $f(x) = \frac{1}{x}$, $x \neq 0$ is continuous

Let 'c' be any non-zero real number, then $\lim_{x\to c} f(x) = \lim_{x\to c} \frac{1}{x} = \frac{1}{c}$. For $c = 0, f(c) = \frac{1}{c}$ So $\lim_{x\to c} f(x) = f(c)$

and hence f is continuous at every point in the domain of f.

Algebra of continuous functions

Suppose f and g are two real functions continuous at a real number c, then, $f+g$, $f-g$, $f.g$ and $\frac{f}{g}$ are continuous at $x=c[g(c) \neq 0]$.

Differentiability

Suppose f is a real function and c is a point in its domain. The derivative of f at c is $f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$

Every differentiable function is continuous, but the converse is not true.

Chain Rule

If $y=f(u)$, where $u=g(x)$ and if both $\frac{dy}{du}, \frac{du}{dx}$ exists, then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$.

Logarithmic differentiation

Let $y=f(x)=[u(x)]^{v(x)}$

$\log y = v(x)\log[u(x)]$

$\frac{1}{y}.\frac{dy}{dx} = v(x)\frac{1}{u(x)}u'(x) + v'(x)\log[u(x)]$

$\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)}u'(x) + v'(x)\log[u(x)]\right]$

e.g. : Let $y = a^x$ Then $\log y = x \log a$

$\frac{1}{y}.\frac{dy}{dx} = \log a$

$\frac{dy}{dx} = y\log a = a^x \log a$.

Derivatives of Implicit functions

If two variables are expressed by some relation then one will be the implicit function of other, is called Implicit function.

For example: Let $y = \cos x - \sin y$, then $\frac{dy}{dx} = \frac{d}{dx}\cos x - \frac{d}{dx}\sin y$

or, $\frac{dy}{dx} = -\sin x - \cos y.\frac{dy}{dx}$ or, $\frac{dy}{dx} = -\sin x/(1+\cos y)$, where $y \neq (2n+1)\pi$

Derivatives of functions in parametric form

Let $x = f(t)$, $y = g(t)$ be two functions of parameter 't'.

Then, $\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$ or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Thus, $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$ (provided $f'(t) \neq 0$)

eg : if $x = a\cos\theta$, $y = a\sin\theta$ then $\frac{dx}{d\theta} = -a\sin\theta$ and $\frac{dy}{d\theta} = a\cos\theta$, and so $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{a\cos\theta}{a\sin\theta} = -\cot\theta$.

Second order derivative

Let $y = f(x)$ then $\frac{dy}{dx} = f'(x)$, if $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}f'(x)$ i.e.,

$\frac{d^2y}{dx^2} = f''(x)$ is the second order derivative of y w.r.t.x.

eg : if $y = 3x^2+2$, then $y' = 6x$ and $y'' = 6$.

Some Standard derivatives

(i) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

(iv) $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

(vi) $\frac{d}{dx}(\text{cosec}^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$

(vii) $\frac{d}{dx}(e^x) = e^x$

(viii) $\frac{d}{dx}(\log x) = \frac{1}{x}$

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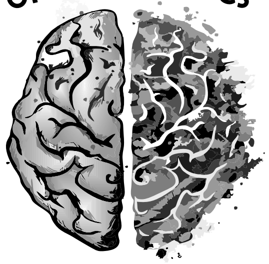
Applications of Derivatives



Maxima and Minima

A point C in the domain of 'f' at which either $f'(C)=0$ or is not differentiable is called a critical point of f.

First derivative test

Let f be continuous at a critical point C in open interval. Then
- **(i)** If $f'(x) > 0$ at every point left of C and $f'(x) < 0$ at every point right of C, then 'C' is a point of local maxima.
- **(ii)** If $f'(x) < 0$ at every point left of C and $f'(x) > 0$ at every point right of C, then 'C' is a point of local minima.
- **(iii)** If $f'(x)$ does not change sign as 'x' increases through C, then 'C' is called the point of inflection.

Second derivative test

Let f be a function defined on given interval, f is twice differentiable at C. Then
- **(i)** $x=C$ is a point of local maxima If $f'(C)=0$ and $f''(C)<0$, $f(C)$ is local maxima of f.
- **(ii)** $x=C$ is a point of local minima if $f'(C)=0$ and $f''(C)>0$. $f(C)$ is local minima of f.
- **(iii)** The test fails if $f'(C)=0$ and $f''(C)=0$

Rate of change of bodies

The change in quantity w.r.t. time is known as rate of change. If a quantity y varies w.r.t. another quantity x, satisfying $y = f(x)$, then $\frac{dy}{dx}$ represents rate of change of y w.r.t. 'x'

Increasing and decreasing functions

If $f'(x) \geq 0 \; \forall \; x \in (a,b)$ then f is increasing in (a,b) and if $f'(x) \leq 0 \; \forall \; x \in (a,b)$, then f is decreasing in (a,b)

eg: Let $f(x) = x^3 - 3x^2 + 4x$, $x \in R$, then

$f'(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1 > 0 \; \forall \; x \in$ R.

So, the function f is strictly increasing on R.

A function f is said to be
- **(i)** increasing on (a,b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) \leq f(x_2) \; \forall \; x_1, x_2 \in (a,b)$, and
- **(ii)** decreasing on (a, b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) \geq f(x_2) \; \forall \; x_1, x_2 \in (a,b)$

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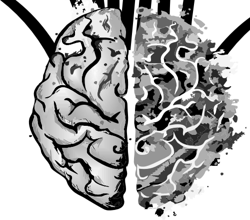
Integrals



Integration

It is the inverse of differentiation. Let, $\frac{d}{dx}F(x)=f(x)$. Then, $\int f(x)dx=F(x)+c$, 'c': constant of integral. These integrals are called indefinite or general integrals. Properties of indefinite integrals are

(i) $\int[f(x)\pm g(x)]\,dx=\int f(x)dx\pm\int g(x)dx$, **(ii)** $\int kf(x)dx=k\int f(x)dx$,

eg : $\int(3x^2+2x)dx=x^3+x^2+c$, where c is real.

Example

$$\int_{-\pi/4}^{\pi/4}\sin^2 x\,dx=2\int_0^{\pi/4}\sin^2 x\,dx=2\int_0^{\pi/4}\left(\frac{1-\cos 2x}{2}\right)dx=\int_0^{\pi/4}(1-\cos 2x)dx=\left[x-\frac{\sin 2x}{2}\right]_0^{\pi/4}=\frac{\pi}{4}-\frac{1}{2}$$

Some standard integrals

(i) $\int x^n dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$ like, $\int dx=x+c$

(ii) $\int\cos x\,dx=\sin x+c$ (iii) $\int\sin x\,dx=-\cos x+c$

(iv) $\int\sec^2 x\,dx=\tan x+c$ (v) $\int\text{cosec}^2 x\,dx=-\cot x+c$

(vi) $\int\sec x\tan x\,dx=\sec x+c$ (vii) $\int\text{cosec}\,x\cot x\,dx=-\text{cosec}\,x+c$

(viii) $\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+c$ (ix) $\int\frac{dx}{\sqrt{1-x^2}}=-\cos^{-1}x+c$

(x) $\int\frac{dx}{1+x^2}=\tan^{-1}x+c$ (xi) $\int\frac{dx}{1+x^2}=-\cot^{-1}x+c$

(xii) $\int e^x dx=e^x+c$ (xiii) $\int a^x dx=\frac{a^x}{\log a}+c$

(xiv) $\int\frac{dx}{x\sqrt{x^2-1}}=\sec^{-1}x+c$ (xv) $\int\frac{dx}{x\sqrt{x^2-1}}=-\text{cosec}^{-1}x+c$

(xvi) $\int\frac{1}{x}dx=\log|x|+c$

Integration by substitution

The method in which we change the variable to some other variable is called the method of substitution. Below problems can be solved by substitution.

$\int\tan x\,dx=\log|\sec x|+c$ $\int\cot x\,dx=\log|\sin x|+c$

$\int\sec x\,dx=\log|\sec x+\tan x|+c$ $\int\text{cosec}\,x\,dx=\log|\text{cosec}\,x-\cot x|+c.$

Integration of some special functions

(i) $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$ (ii) $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$

(iii) $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$ (iv) $\int\frac{dx}{\sqrt{x^2-a^2}}=\log\left|x+\sqrt{x^2-a^2}\right|+c$

(v) $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+c$ (vi) $\int\frac{dx}{\sqrt{x^2+a^2}}=\log\left|x+\sqrt{x^2+a^2}\right|+c.$

(vii) $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c.$

(viii) $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c.$

(ix) $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c.$

Integration by partial fractions

A rational function of the form $\frac{P(x)}{Q(x)}[Q(x)\neq 0]=T(x)+\frac{P_1(x)}{Q(x)}$, $P_1(x)$ has degree less than that of $Q(x)$. We can integrate $\frac{P_1(x)}{Q(x)}$ by expressing it in the following forms –

(i) $\frac{px+q}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}, a,b\neq 0.$

(ii) $\frac{px+q}{(x-a)^2}=\frac{A}{x-a}+\frac{B}{(x-a)^2}$

(iii) $\frac{px^2+qx+r}{(x-a)^2(x-b)}=\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$

(iv) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}=\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

(v) $\frac{Px+q}{ax^2+bx+c}=\frac{A\frac{d}{dx}(ax^2+bx+c)}{ax^2+bx+c}+\frac{B}{ax^2+bx+c}$

Integration by parts

$$\int f_1(x)f_2(x)dx=f_1(x)\int f_2(x)dx-\int\left[\frac{d}{dx}f_1(x)\int f_2(x)dx\right]dx$$

First fundamental theorem of integral calculus

Let the area function be defined by $A(x)=\int_a^x f(x)dx\,\forall x\geq a$, where f is continuous on $[a,b]$ then $A'(x)=f(x)\forall x\in[a,b]$.

Second fundamental theorem of integral calculus

Let f be a continuous function of x defined on $[a, b]$ and let F be another function such that $\frac{d}{dx}F(x)=f(x)\forall x\in$ domain of f_1 then $\int_a^b f(x)dx=[F(x)+c]_a^b=F(b)-F(a)$. This is called the definite integral of f over the range $[a, b]$, where a and b are called the limits of integration, a being the lower limit and b be the upper limit.

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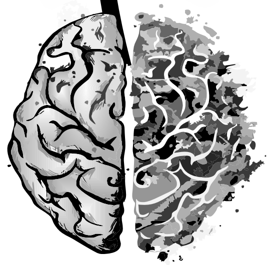
Applications of the Integrals



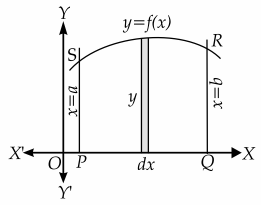
Area under simple curves

The area of the region bounded by the curve $y = f(x)$, *X- axis* and the lines $x = a$ and $x = b\,(b > a)$ is given by

$$A = \int_a^b y\,dx \text{ or } \int_a^b f(x)dx$$

eg : The area bounded by $y = x^2$, X- axis in I quadrant and the lines $x = 2$ and $x = 3$ is -

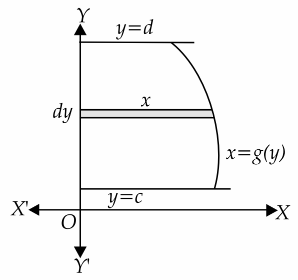
$$A = \int_2^3 y\,dx = \int_2^3 x^2\,dx = \left[\frac{x^3}{3}\right]_2^3 = \frac{1}{3}(27-8) = \frac{19}{3} \text{ Sq.units.}$$



The area of the region bounded by the curve $x = f(y)$, $Y-axis$ and the lines $y = c$ and $y = d\,(d > c)$ is given by $A = \int_c^d x\,dy$ or $\int_c^d f(y)dy$

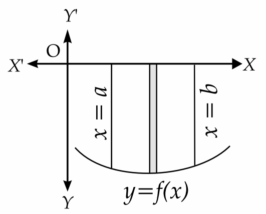
eg : The area bounded by $x = y^3$, $Y-axis$ in the I quadrant and the lines $y = 1$ and $y = 2$ is

$$\int_a^b x dy = \int_1^2 y^3\,dy = \left[\frac{1}{4}y^4\right]_1^2 = \frac{1}{4}(2^4 - 1^4) = \frac{15}{4} \text{ Sq. units}$$



If the curve under consideration lies below X-axis, then $f(x) < 0$ from $x = a$ to $x = b$, the area bounded by the curve $y = f(x)$ and the ordinates $x = a$, $x = b$ and X-axis is negative

$$A = \left|\int_a^b f(x)dx\right|$$



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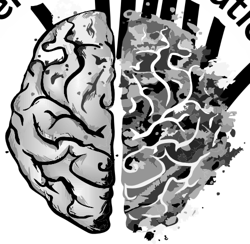
Differential Equations



Definition

An equation involving derivatives of the dependent variable with respect to independent variable (variables) is called a differential equation. If there is only one independent variable, then we call it as an ordinary differential equation. eg: $2\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)=0$.

Order of a Differential Equation

It is the order of the highest order derivative occurring in the differential equation

eg: The order of $\frac{dy}{dx}=e^x$ is one and order of $\frac{d^2y}{dx^2}+x=0$ is two.

Degree of a Differential Equation

It is defined if the differential equation is a polynomial equation in its derivatives, and is defined as the highest power (positive integer only) of the highest order derivative.

eg: The degree of $\left(\frac{d^2y}{dx^2}\right)^3+\frac{dy}{dx}=0$ is three

Order and degree (if defined) of a differential equation are always positive integers.

Solution of a Differential Equation

A function which satisfies the given differential equation is called its solution. The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution and the solution free from arbitrary constants is called particular solution.

eg: $y=e^x+1$ is a solution of $y''-y'=0$.

Since $y'=e^x$ and $y''=e^x \Rightarrow y''-y'=e^x-e^x=0$.

The order of a differential equation representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves.

eg: Let the family of curves be

$y=mx$, $m=$ constant, then, $y'=m$

$y=y'x \Rightarrow y=\frac{dy}{dx}x \Rightarrow x\frac{dy}{dx}-y=0$.

Homogeneous Differential Equations

A differential equation which can be expressed in the form $\frac{dy}{dx}=f(x,y)$ or $\frac{dx}{dy}=g(x,y)$, where, $f(x,y)$ and $g(x,y)$ are homogeneous functions of degree zero is called a homogenous differential equation

eg : $(x^2+xy)dy=(x^2+y^2)dx$

To solve this, we substitute $y=vx$, and $\frac{dy}{dx}=v+x\frac{dv}{dx}$

Linear Differential Equations

The differential equation of the form $\frac{dy}{dx}+Py=Q$, where P, Q are constants or functions of 'x' only is called a first order linear differential equation. Its solution is given as $ye^{\int P.dx}=\int Q.e^{\int P.dx}\,dx+c$. $eg:\frac{dy}{dx}+3y=2x$ has

solution $ye^{\int 3.dx}=\int 2x.e^{\int 3.dx}dx+c \Rightarrow ye^{3x}=2\int xe^{3x}dx+c$.

Variable Separation Method

It is used to solve such an equation in which variables can be separated completely.

eg : $ydx = xdy$ can be solved as $\frac{dx}{x}=\frac{dy}{y}$;

Integrating both sides

$\log x=\log y+\log c \Rightarrow \frac{x}{y}=c \Rightarrow x=cy$,

is the solution.

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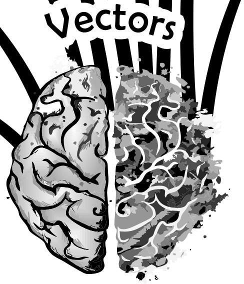
Vectors



Unit vector

For a given vector $\vec{a}$, the vector $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ gives the unit vector in the direction of $\vec{a}$. e.g., If $\vec{a} = 5\hat{i}$, then $\hat{a} = \frac{5\hat{i}}{|5|} = \hat{i}$, which is a unit vector.

Direction ratios and direction cosines

The scalar components of a vector are its direction ratios, and represent its projections along the respective axes. The magnitude (r) direction ratios (a, b, c) and direction cosines (l,m,n) of vector $a\hat{i} + b\hat{j} + c\hat{k}$ are related as:

$l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r}$ Where $r = |\sqrt{a^2 + b^2 + c^2}|$

e.g. : If $\overrightarrow{AB} = \hat{i} + 2\hat{j} + 3\hat{k}$, then $r = |\sqrt{1+4+9}| = \sqrt{14}$

Direction ratios are (1, 2, 3) and direction cosines are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

Position Vector

The position vector of a point R dividing a line segment joining P, Q whose position vectors are $\vec{a}, \vec{b}$ resp., in the ratio $m : n$.

(i) internally is $\frac{n\vec{a} + m\vec{b}}{m+n}$, (ii) externally is $\frac{m\vec{b} - n\vec{a}}{m-n}$

Vector

A quantity that has both magnitude and direction is called a vector. The distance between the initial and terminal points of a vector is called its magnitude. Magnitude of vector $\overrightarrow{AB}$ is $|\overrightarrow{AB}|$.

Scalar product of two vectors

If $\vec{a}, \vec{b}$ are the vectors and 'θ', angle between them, then their scalar product $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

Cross product of two vectors

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, $\hat{n}$ is a unit vector perpendicular to line joining a, b.

Position of vectors

Position vector of a point P (x, y, z) is $x\hat{i} + y\hat{j} + z\hat{k}$ and its magnitude is $OP(r) = \sqrt{x^2 + y^2 + z^2}$. eg: Position vector of P (2, 3, 5) is $2\hat{i} + 3\hat{j} + 5\hat{k}$ and its magnitude is $\sqrt{2^2 + 3^2 + 5^2} = \sqrt{38}$.

Types of Vectors

(i) Zero vector (initial and terminal points coincide)
(ii) Unit vector (magnitude is unity)
(iii) Coinitial vectors (same initial points)
(iv) Collinear vectors (parallel to the same line)
(v) Equal vectors (same magnitude and direction)
(vi) Negative of a vector (same magnitude, opp. direction)

Properties of Vector

If we have two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and λ is any scalar, then-

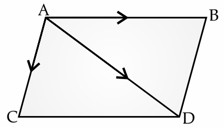
(i) $\vec{a} \pm \vec{b} = (a_1 \pm b_1)\hat{i} + (a_2 \pm b_2)\hat{j} + (a_3 \pm b_3)\hat{k}$

(ii) $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

(iii) $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ and

(iv) $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

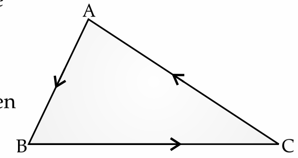
The vector sum of two co-initials vectors is given by the diagonal of the parallelogram whose adjacent sides are given vectors.



If $\overrightarrow{AB}, \overrightarrow{AC}$ are the given vectors, then $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$

The vector sum of the three sides of a triangle taken in order is $\vec{0}$. i.e.,

If ABC is given triangle, then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$.



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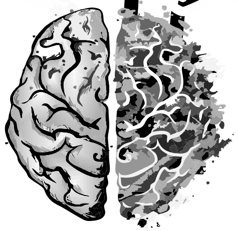
Three Dimensional Geometry



Direction ratios and direction cosines of a line

D. Cs of a line are the cosines of the angles made by the line with the positive direction of the co-ordinate axes. If l, m, n are the D. Cs of a line, then $l^2 + m^2 + n^2 = 1$. D. Cs of a line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}$, where $PQ=\left|\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\right|$

D.Rs of a line are the no.s which are proportional to the D.Cs of the line if l, m, n are the D.Cs and a, b, c are D.Rs of a line, then

$$l=\frac{a}{\left|\sqrt{a^2+b^2+c^2}\right|}, m=\frac{b}{\left|\sqrt{a^2+b^2+c^2}\right|}, n=\frac{c}{\left|\sqrt{a^2+b^2+c^2}\right|}$$

Parallel lines

(i) Two skew lines are the line segment perpendicular to both the lines

(ii) $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ is $\frac{\left|(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)\right|}{\left|(\vec{b}_1\times\vec{b}_2)\right|}$

(iii) $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is

$$=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

(iv) Distance between parallel lines $\vec{r}=\vec{a}_1+\lambda\vec{b}$ and $\vec{r}=\vec{a}_2+\mu\vec{b}$ is $\frac{\left|\vec{b}\times(\vec{a}_2-\vec{a}_1)\right|}{\left|\vec{b}\right|}$

Skew lines

These are the lines in space which are neither parallel nor intersecting. They lie in different planes. Angle between skew lines is the angle between two intersecting lines drawn from any point (origin) parallel to each of the skew lines.

Angle between the two lines

If $l_1, m_1, n_1, l_2, m_2, n_2$ are the D.Cs and $a_1, b_1, c_1, a_2, b_2, c_2$ are the D.Rs of the two lines and 'θ' is the acute angle between them, then

$$\cos\theta=|l_1l_2+m_1m_2+n_1n_2|=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

Equation of line Vector in 3D

Vector equation of a line passing through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\vec{b}$

Vector equation of a line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$

Equation of a line through point (x_1, y_1, z_1) and having D.Cs l, m, n is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ Also, equation of a line that passes through two points is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

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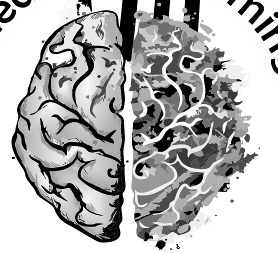
Linear Programming



Definition

A L.P.P. is one that is concerned with finding the optimal value (max. or min.) of a linear function of several variables (called objective function) subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints). Variables are sometimes called decision variables and are non-negative.

Solution of a L.P.P.

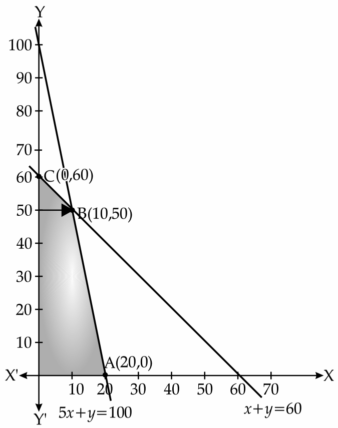
The common region determined by all the constraints including the non-negative constraint $x \geq 0, y \geq 0$ of a L.P.P. is called the feasible region (or solution region) for the problem. Points within and on the boundary of the feasible region represent feasible solutions of the constraints. Any point outside the feasible region is an infeasible solution. Any point in the feasible region that gives the optimal value (max. or min.) of the objective function is called an optimal solution.

e.g., : Max $Z = 250x + 75y$, subject to the

Constraints: $5x + y \leq 100$

$x + y \leq 60$

$x \geq 0, y \geq 0$ is an L.P.P.



Fundamental Theorems

Theorem 1 : Let R be the feasible region (convex polygon) for a L.P.P. and let $Z=ax+by$ be the objective function. When Z has an optimal value (max. or min.), where the variables x,y are subject to the constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region,

Theorem 2: Let R be the feasible region for a L.P.P. and let $Z=ax+by$ be the objective function. If R is bounded then the objective function Z has both a max. and a min. value on R and each of these occurs at a corner point (vertex) of R. If the feasible region is unbounded, then a max. or a min. may not exist. If it exists, it must occur at a corner point of R.

Corner point method

1. Find the feasible region of the linear programming problem and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.
2. Evaluate the objective function $Z = ax+by$ at each corner point. Let M and m, respectively denote the largest and smallest values of these points.
3. (i) When the feasible region is **bounded,** M and m are the maximum and minimum values of Z

   (ii) In case, the feasible region is **unbounded,** we have:

   (a) M is the maximum value of Z, if the open half plane determined by $ax+by>M$ has no point in common with the feasible region. Otherwise, Z has no maximum value.

   (b) Similarly, m is the minimum value of Z, if the open half plane determined by $ax+by<m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

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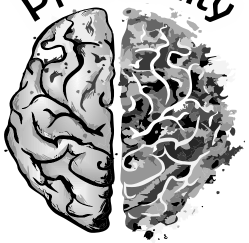
Probability



Conditional Probability

The probability of the event E is called the conditional probability of E given that F has already occurred, and is denoted by $P(E/F)$. Also,

$$P(E/F)=\frac{P(E\cap F)}{P(F)}, P(F)\neq 0.$$

Properties

(i) $0\leq P(E/F)\leq 1$, $P(E'/F)=1-P(E/F)$

(ii) $P((E\cup F)/G)=P(E/G)+P(F/G)-P((E\cap F)/G)$

(iii) $P(E\cap F)=P(E)P(F/E), P(E)\neq 0$

(iv) $P(F\cap E)=P(F)P(E/F), P(F)\neq 0$

e.g., : if $P(A)=\frac{7}{13}$, $P(B)=\frac{9}{13}$ and $P(A\cap B)=\frac{4}{13}$, then

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{4}{13}}{\frac{9}{13}}=\frac{4}{9}$$

Independent Event

If E and F are independent, then

$P(E\cap F)=P(E)P(F)$, $P(E-F)=P(E)$, $P(F)\neq 0$

and $P(F-E)=P(F)$, $P(E)\neq 0$.

If A,B,C are mutually independent events then

(i) $P(A\cap B)=P(A).P(B)$

(ii) $P(A\cap C)=P(A).P(C)$

(iii) $P(B\cap C)=P(B).P(C)$

(iv) $P(A\cap B\cap C)=P(A).P(B).P(C)$

Theorem of total probability

Let $\{E_1, E_2 \ldots.E_n\}$ be a partition of a sample space 'S' and suppose that each of $E_1, E_2, \ldots.E_n$ has non-zero probability. Let 'A' be any event associated with S, then $P(A) = P(E_1) P(A/E_1) + P(E_2) P(A/E_2) + \ldots.+ P(E_n) P(A/E_n)$.

$$P\left(\frac{E_i}{A}\right) = \sum_{i=1}^{n} P(E_i)P(A/E_i)$$

Probability Distribution

The probability distribution of a random variable x is the system of numbers

$x: x_1\, x_2 \ldots\ldots\, x_n$, $P(x): p_1\, p_2 \ldots\ldots\, p_n$

where , $p_i>0$,

$$\sum_{i=1}^{n} p_i = 1, i=1,2,\ldots..,n.$$

Bayes' Theorem

If $E_1, E_2, \ldots., E_n$ are events which constitute a partition of sample space S, i.e., $E_1, E_2, \ldots E_n$ are pairwise disjoint and $E_1\cup E_2\cup\ldots.\cup E_n=S$ and A be any event with non-zero probability,

then $P(E_i/A)=\dfrac{P(E_i)P(A/E_i)}{\sum_{i=1}^{n}P(E_i)P(A/E_i)}, n=1,2,3\ldots\ldots..n$

Mean of a random variable

Let x be a random variable whose possible values $x_1, x_2, \ldots., x_n$ occur with probabilities $p_1, p_2, \ldots., p_n$ respectively. Then, mean of x is, $\mu=\sum_{i=1}^{n} x_i p_i$. It is also called the expectation of x, denoted by $E(x)$.

Random Variable

Real valued function whose domain is the sample space of a random experiment.

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